Categorical symmetry
– a holographic view of symmetry

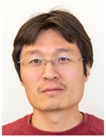
Xiao-Gang Wen (MIT), June, 2020

Kong-Wen arXiv:1405.5858

Ji-Wen arXiv:1912.13492

Kong-Lan-Wen-Zhang-Zheng arXiv:2003.08898; arXiv:2005.14178

Ji



Kong

Lan

Zhang

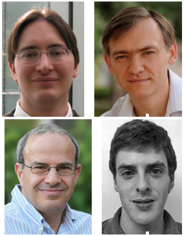
Zheng

Simons Collaboration on
Ultra-Quantum Matter

Symmetry in quantum systems

- What is a **symmetry?**

- A **bosonic symmetry** is a set of linear constraints on local Hamiltonians of **bosonic systems** (*ie* qubit systems).
- A **fermionic symmetry** is a set of linear constraints on local Hamiltonians of **fermionic systems**.
- **Linear constraints**: $W_a H = H W_a$, where a labels different symmetry transformations. It can be (1) different group elements, and (2) different loops, close surfaces, *etc* where the operator act, ...
- If the transformation acts on the whole space $\to$ the usual global symmetry (0-symmetry).
- If the trans. acts on all the codimension-k closed sub-spaces $\to$ the k-symmetry.

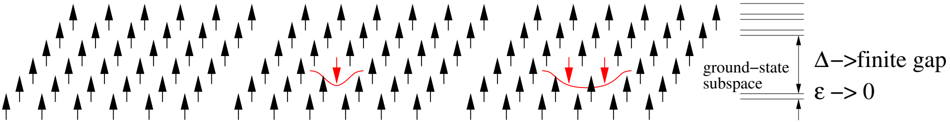


Gaiotto-Kapustin-Seiberg-Willett arXiv:1412.5148

- **Symmetry** = a class of Hamiltonians $\{H \mid W_a H = H W_a\}$, which are called the **symmetric Hamiltonians**

View symmetry via their **charged excitations**

- A **quantum system** (a 2d spin-1/2 system) is determined by **Hamiltonian** $H = -\sum_i Z_i$ and **deformation class** $\{\delta H\}$.



- **An excitation** = something can be trapped: H has a gap $H + \delta H_{\text{trap}}$ also has a gap → trapped states: $|\downarrow\rangle$, $|\downarrow\downarrow\rangle$, ...

- **Type**: *(determined by H and deformation class $\{\delta H\}$)*

- Without symmetry *($\{\delta H\}$ are formed by local operators)*
$|\downarrow\rangle$, $|\downarrow\downarrow\rangle$ can be deformed into $|0\rangle$ without closing the gap.
$|\downarrow\rangle \sim |\downarrow\downarrow\rangle \sim |0\rangle$ the same trivial type **1**.

- With Z_2-symm. $U = \prod_i Z_i$ $(\{\delta H \mid \delta H U = U \delta H\})$
$|\downarrow\downarrow\rangle$ and $|0\rangle$ are of the same trivial type **1** *(Z_2-charge-0)*.
$|\downarrow\rangle$ has a different type e *(Z_2-charge-1)*.

The charge excitations form a fusion 2-category

- With the Z_2-symmetry, the excitations have a fusion rule
  $\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$, $\mathbf{1} \otimes e = e \otimes \mathbf{1} = e$, $\mathbf{1}$ is the fusion unit
  $e \otimes e = \mathbf{1}$, e has mod 2 conservation

\- $\{\mathbf{1}, e\}$ generate a fusion 2-category $2\mathcal{R}\mathrm{ep}(Z_2)$.

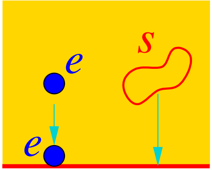
- In 2-dimensional space, we have e_s = *e-condensed string*
  string-like excitations $\to$ objects ($\{\mathbf{1}_s, e_s\}$ *trivial, descendent*),
  point-like excitations $\to$ 1-morphisms ($\{\mathbf{1}, e\}$ *generators*).
  2 layers $\to$ a fusion 2-category. Kong-Wen-Zhen arXiv:1502.01690
  Gaiotto & Johnson-Freyd arXiv:1905.09566

  Tannaka duality: symmetric fusion category $\leftrightarrow$ symm. group

- **Categorical view of symmetry**:
  The class of 2d Z_2-symmetric Hamiltonians: $\{H_{Z_2}\}$
  $=$ The class of fusion-preserveing Hamiltonians describing
  interacting excitations in $2\mathcal{R}\mathrm{ep}(Z_2)$: $\{H_{2\mathcal{R}\mathrm{ep}(Z_2)}\}$.
  (The class of Hamiltonians describing particles with mod 2 conservation. No longer think about symm. transformations)

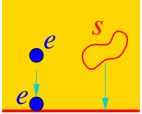
Use entanglement to simulate symmetry

- **Z_2-symmetry in 2d = Fusion 2-category** $2\mathcal{R}\mathrm{ep}(Z_2)$
- $2\mathcal{R}\mathrm{ep}(Z_2)$ describes topo. excitations from a topo. order without symm. *But which topo. order has excitations* $\{\mathbf{1}, e\}$*?*
- The fusion 2-category $\{\mathbf{1}, e\} = 2\mathcal{R}\mathrm{ep}(Z_2)$ describes the excitations on a boundary of **3d Z_2 gauge theory**
- The excitations in 3d Z_2-gauge theory is generated by point-like excitations e (the bosonic Z_2 charge) and string-like excitations s (the bosonic Z_2-flux string)
- Bulk fusion rule: $e \otimes e = \mathbf{1}, \quad s \otimes s = \mathbf{1}_s$ (trivial string)
- The boundary is induced by the Z_2-flux loops condensation. The boundary excitations are described by $\{\mathbf{1}, e\} = 2\mathcal{R}\mathrm{ep}(Z_2)$.
  Entanglement (topological order) in one higher dimension → symmetry



- The class of 2d Z_2-symm. Hamiltonians: $\{H_{Z_2}\}$ = Boundary Hamiltonians of 3d Z_2 gauge theory (∞ bulk gap): $\{H^{\text{bndry}}_{Z_2\text{-gauge}}\}$

Boundary symmetry from bulk symmetry



- The mod 2 conservation of the bulk-e $\to$ The Z_2 0-symmetry in the bulk $\to$ The mod 2 conservation of boundary-e $\to$ The Z_2 symmetry at the boundary.

$\{\mathbf{1}, e\} = 2\mathcal{R}\text{ep}(Z_2)$

String condensed boundary = trivial Z_2-symmetric phase

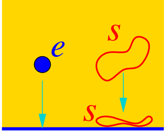
Z_2 0-symmetry is not broken, $Z_2^{(1)}$ 1-symmetry is broken.

- But the bulk actually has a bigger symmetry – $Z_2 \times Z_2^{(1)}$ **Categorical symmetry**. The mod 2 conservation of the bulk-s $\to$ The $Z_2^{(1)}$ 1-symmetry in the bulk.

Charge condensed boundary = Z_2-symm. breaking phase

Z_2 0-symmetry is broken, $Z_2^{(1)}$ 1-symmetry is not broken.

- The Z_2-symmetry breaking phase has excitations $\{\mathbf{1}, s\} = 2\mathcal{V}\text{ec}_{Z_2}$. ($s$ is the string-like symmetry breaking domain walls.) The 3d Z_2 gauge theory has a Z_2-charge condensed boundary with excitations $\{\mathbf{1}, s\} = 2\mathcal{V}\text{ec}_{Z_2}$.



$\{\mathbf{1}, s\} = 2\mathcal{V}\text{ec}_{Z_2}$

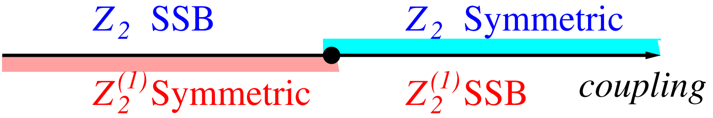
$Z_2 \vee Z_2^{(1)}$ categorical symmetry = holographic view

- The categorical symmetry is not quite $Z_2 \times Z_2^{(1)}$. The particle e and the string s in bulk have a non-trivial mutual statistics. So we denote the categorical symmetry as $Z_2 \vee Z_2^{(1)}$.
- **A 2d Z_2-symmetric system actually have a bigger symmetry – the $Z_2 \vee Z_2^{(1)}$ Categorical symmetry**.
  The class of 2d Z_2-symmetric Hamiltonians: $\{H_{Z_2}\}$
  $=$ The Hamiltonians for excitations $2\mathcal{R}\mathrm{ep}(Z_2)$: $\{H_{2\mathcal{R}\mathrm{ep}(Z_2)}\}$
  $=$ Boundary of 3d Z_2 gauge theory (∞ bulk gap): $\{H^{\text{bndry}}_{Z_2\text{-gauge}}\}$
  $=$ The class of Hamiltonians with $Z_2 \vee Z_2^{(1)}$ symm. $\{H^{\text{bndry}}_{Z_2\text{-gauge}}\}$
- All phases of Z_2-symmetric systems
  $=$ All phases of $Z_2 \vee Z_2^{(1)}$-symmetric systems
  All boundary phase of 3+1D Z_2 gauge theory w/ ∞ bulk gap.

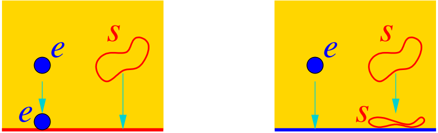
$Z_2 \vee Z_2^{(1)}$ categorical symmetry at critical point

- A **holographic view** on symmetry breaking transition:
  A Z_2 symmetric system actually has a bigger $Z_2 \vee Z_2^{(1)}$ categorical symmetry

Ji-Wen arXiv:1912.13492

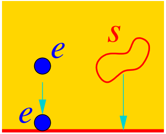


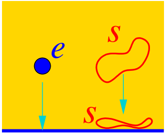
- **The critical point at the symmetry-breaking transition has the full (unbroken) $Z_2 \vee Z_2^{(1)}$ categorical symmetry.**



The critical point = The minimal non-condensing boundary of the 3+1D Z_2 guage theory

States with full categorical symm. must be gapless





For a 2d system with $Z_2 \vee Z_2^{(1)}$ categorical symmetry:

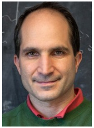
- its gapped phases must either break the Z_2 0-symmetry or the $Z_2^{(1)}$ 1-symmetry, but not both.
- The phase with the full $Z_2 \vee Z_2^{(1)}$ categorical symmetry must be gapless.

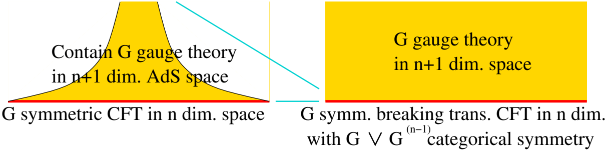
(The 1d version of the above results were proved by Levin arXiv:1903.09028)

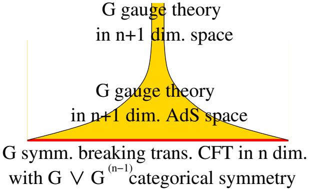
Apply categorical symmetry to AdS/CFT duality

Maldacena arXiv:hep-th/9711200; Witten arXiv:hep-th/9802150

- **Witten**: "for gauge theory, suppose the AdS theory has a gauge group G, [...] Then in the scenario of [13], the group G is a global symmetry group of the conformal field theory on the boundary."



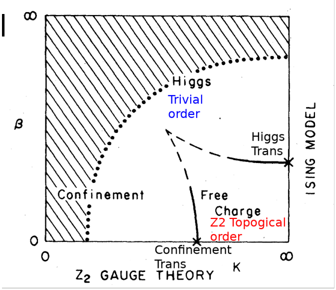


- G-symm.-breaking-transition CFT has a categorical symm. described by the G-gauge theroy in one higher dimension, *which uniquely determines the bulk theory* $\to$ A proposal: Pure G-gauge theory (w/ charge fluc. & gravity) in $(n+1)$-dim. AdS space = CFT at the G-symm.-breaking-transition in n-dim. space, not other CFT's with G-symmetry.

Ji-Wen arXiv:1912.13492

Apply categorical symm. to 3+1D Z_2 gauge theory

- Phase transitions from Z_2 topological order (described by Z_2 guage theory) to trivial product state:
- Higgs transition induced by Z_2 point-charge condensation. *(same as 3+1D Ising transition)* The Ising CFT has a Z_2 symmetry.
- Confinement transition induced by Z_2 flux-string condensation. The transition is first order → a critical point. The confinement CFT has a $Z^{(1)}$ symmetry.



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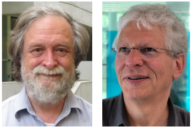

- *The Ising CFT = The confinement CFT ???* They might have the same emergent symmetries.
- The Ising CFT has categorical symmetry $Z_2 \vee Z_2^{(2)}$
  The confinement CFT has categorical symmetry $Z_2^{(1)} \vee Z_2^{(1)}$

Ji-Wen arXiv:1912.13492

Categorical symmetry = Gravitational anomaly

Symmetry G selects a class of symmetric Hamiltonians $\{H_G\}$.

- Gravitational anomaly (denoed by M) also selects a class of (effective) Hamiltonians $\{H_{\mathrm{M}}\}$ that have the same anomaly M.

- $\{H_{\mathrm{M}}\}$ is a linear space: $H_{\mathrm{M}} + H'_{\mathrm{M}} \in \{H_{\mathrm{M}}\}$.
- Gravitational anomaly is also a set on linear constraints, just like symmetry.

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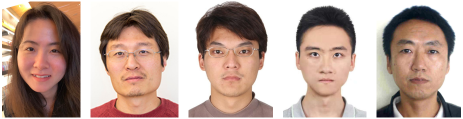
Kong-Lan-Wen-Zhang-Zheng arXiv:2003.08898; arXiv:2005.14178



Every 0-symmetry, higher symmetry, or even the most general algebraic higher symmetry corresponds to a gravitational anomaly. *(Many-to-one correspondence)*
Due to this connection, we also refer gravitational anomaly as **categorical symmetry**.

- Some categorical symmetries (gravitational anomalies) may not correspond to a symm., not even the algebraic higher symm.

What is gravitational anomaly?

- **Traditionally**, an gravitational anomaly is a non-invariance of the path integral measure under a diffeomorphism transformation of the spacetime.
- **Recent point of view**: a fusion n-category $\mathcal{C}$ describing the excitations in n-dimensional space (*ie* a field theory) has a gravitational anomaly if $\mathcal{C}$ cannot be realized by excitations in a gapped phase of a bosonic lattice model in the same dimension and without symmetry.
- **Categorical view of symmetry**: We use fusion 2-category $2\mathcal{R}\mathrm{ep}(Z_2) = \{\mathbf{1}, e\}$ to describe 2d Z_2 symmetry.
  ie we ignore the symmetry, and use fusion 2-category $2\mathcal{R}\mathrm{ep}(Z_2) = \{\mathbf{1}, e\}$ *and their arbitary interactions (preserve fusion) to obtain the class of Hamiltonians* $\{H_{2\mathcal{R}\mathrm{ep}(Z_2)}\}$
  $\to$ *the class of Hamiltonians selected by the* Z_2 *symm.* $\{H_{Z_2}\}$.
- $2\mathcal{R}\mathrm{ep}(Z_2) = \{\mathbf{1}, e\}$ has gravitational anomaly, since it cannot be realized by 2d bosonic lattice model without symmetry.

What REALY is gravitational anomaly?

- **A conjecture**: A fusion n-category $\mathcal{C}$ can always be realized by the excitations on a certain boundary of a bosonic lattice model in one higher dimension (*ie* in $(n+1)$-dim. space).
  - An anomaly-free fusion n-category $\mathcal{C}$ can be realized by the excitations on a certain boundary of a product state.
  - An anomalous fusion n-category $\mathcal{C}$ can be realized by the excitations on a certain boundary of a non-trivial anomaly-free topological order M in one higher dimension.

- **Holographic principle** A boundary fusion n-category $\mathcal{C}$ uniquely determines a bulk topological order $\mathrm{M} = \mathrm{bulk}(\mathcal{C})$. Kong-Wen arXiv:1405.5858; Kong-Wen-Zheng arXiv:1702.00673
  Gravitational anomaly of $\mathcal{C}$ = bulk topological order M
  Those anomalies are often non-invertible. *Ji-Wen arXiv:1905.13279*
- Symmetry $(2\mathcal{R}\mathrm{ep}(Z_2)) \to$ Gravitational anomaly = Bulk topo. order (bulk Z_2 gauge theory) = categorical symm. $(Z_2 \vee Z_2^{(1)})$

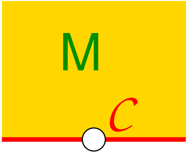
Emergent categorical symmetry

- Consider a **field theory** in n-dimensional space with low energy excitation described by a fusion n-category $\mathcal{C}$.

- We assume higher energy excitations always have very high energies, and ignore them.

- Here **field theory** means a theory whose UV regularization is not specified. *When we say a field theory has a property, we mean that there exists a UV regularizations for the field theory, and such regularized field theory has the property. (There are may be other different UV regularizations where regularized field theory does not the property).*

- Some excitations in $\mathcal{C}$ may come from symmetry charge (e.g. $2\mathcal{R}\mathrm{ep}(Z_2)$), and other are topological. But the distinction is not important, and we pretend all excitations are topological excitations.

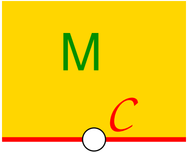


- **The emergent category symmetry in $\mathcal{C}$ is given by a topological order in one higher dimension $\mathrm{M} = \mathrm{bulk}(\mathcal{C})$.**

Emergent categorical symmetry → Low energy properties

The emergent categorical symmetry M controls the low energy properties of those excitations $\mathcal{C}$ (such as their condensations)

- All the gapped and gapless phases formed by the low energy excitations $\mathcal{C}$ have the same categorical symmetry M (*ie* the same gravitational anomaly), which may be partially spontaneously broken.



- **Constraint on possible phases**: The excitations $\mathcal{C}'$, in a gapped phase formed by condensing excitations in $\mathcal{C}$, must satisfy bulk$(\mathcal{C}')$ = bulk$(\mathcal{C})$ → The usual amonaly matching

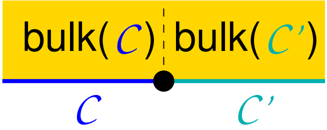
- The new insight here is that we can combine (higher, algebraic higher, 't Hooft anomalous, *etc*) symmetries and (low energy) gravitational anomalies into an **effective gravitational anomaly** → the emergent **categorical symmetry**

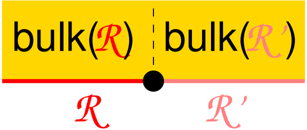
Emergent categorical symmetry → duality relation

- **Duality relation**: two gapped field theories with low energy excitations $\mathcal{C}$ and $\mathcal{C}'$ are **dual-equivalent** if and only if they have the same **categorical symmetry** $\text{bulk}(\mathcal{C}) = \text{bulk}(\mathcal{C}')$.

- **Dual-equivalent**: the states formed by $\mathcal{C}$ have a one-to-one correspondence with the states formed by $\mathcal{C}'$.

- *In $n+1D$, pure $SU(N)$ gauge theory, $Z_N^{(1)}$ bosonic models, $Z_N^{(n-2)}$ bosonic models are dual-equivalent: $\mathrm{M} = Z_N^{(1)} \vee Z_N^{(n-2)}$.*





- **Duality relation**: two symmetries $\mathcal{R}$ and $\mathcal{R}'$ (charge objects) are **dual-equivalent** if and only if they have the same **categorical symmetry** $\text{bulk}(\mathcal{R}) = \text{bulk}(\mathcal{R}')$.

- *An 1d anomalous Z_2^3 symm and D_4 symm are dual-equivalent.*

A theory for most general anomaly-free symmetry

- **Anomaly-free symmetry** allows **trivial symmetric phase** (an unique ground state on closed space of any topology)
- The *set* of **charge objects**, $\mathcal{R}$, of an anomaly-free symmetry is the *set* of **types** of excitations on a *trivial symmetric state*
- The charge objects of a symmetry in n-dimensional space form a fusion n-category, called the **representation category** $\mathcal{R}$.

- For nd bosonic 0-symmetry G, its charge objects (the particles carrying representations of G) generate the so called fusion n-category $\mathcal{R} = n\mathcal{R}\mathrm{ep}(G)$.
- For nd bosonic $(n-1)$-symmetry $Z_2^{(n-1)}$, its charge objects (the $(n-1)$-dimensional domain wall) generate the so called fusion n-category $\mathcal{R} = n\mathcal{V}\mathrm{ec}_{Z_2}$.
- $\mathcal{R} = n\mathcal{V}\mathrm{ec}$ describes excitations on bosonic product state without bosonic symmetry
- $\mathcal{R}_f = n\,\mathrm{s}\mathcal{V}\mathrm{ec} \sim n\mathcal{R}\mathrm{ep}(Z_2)$ describes excitations on fermionic product state without fermionic symmetry

Local fusion n-category $\rightarrow$ algebraic higher symm.

$\mathcal{R}$ is a special kind of fusion n-category. *Which kind?*

- A fusion n-category describes the charge objects of a symmetry if it can be reduced to the fusion n-category of no symmetry (via explicit symmetry breaking) $\rightarrow$ Math definition:

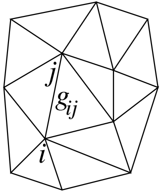
Local fusion category (B: $\beta : \mathcal{R} \overset{\text{top}}{\hookrightarrow} n\mathcal{V}\mathrm{ec}$. F: $\beta : \mathcal{R}_f \overset{\text{top}}{\hookrightarrow} n\,\mathrm{s}\mathcal{V}\mathrm{ec}$) defines **algebraic higher symmetry** (a most general symmetry?)

- **A conjecture**

The class of Hamiltonians for int. excitations in $\mathcal{R}$: $\{H_{\mathcal{R}}\}$
$=$ The class of symmetric Hamiltonians obtained from a set of linear constraints: $\{H_{W_a} \mid W_a H = H W_a\}$

Local fusion categories classify algebraic higher higher symmetries

Algebraic higher symmetry $n\mathcal{V}\mathrm{ec}_G$



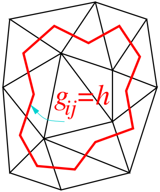
- Consider a G symmetry breaking state in nd. The domain walls are labeled by $h \in G$ with fusion $h \otimes h' = hh'$ → fusion n-category $n\mathcal{V}\mathrm{ec}_G$. *Is $n\mathcal{V}\mathrm{ec}_G$ a local fusion category? Does $n\mathcal{V}\mathrm{ec}_G$ describe a symm?*
- The fusion of the domain walls → a conservation law → a "symmetry". *But the symmetry breaking state is not a product state, and the domain walls may not be the charge objects. Also what is the symmetry transformations?*
- A lattice model on a trianglation of a n-dimensional space. The degrees of freedom on the links ij are given by $g_{ij} \in G$

$$H = -J \sum_{ij} \delta(g_{ij}) - B \sum_i \sum_{h \in G} Q_h(i) - U \sum_{ijk} \delta(g_{ij} g_{jk} g_{ik}^{-1}),$$

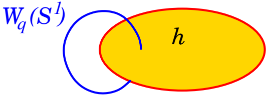
$$Q_h(i)|\cdots, g_{ij}, g_{ki}, \cdots\rangle = |\cdots, hg_{ij}, g_{ki}h^{-1}, \cdots\rangle, \quad g_{ij} = g_{ji}^{-1}.$$

- **Algebraic $(n-1)$-symmetry** is generated by Wilson loop operators $W_q(S^1) = \mathrm{Tr} \prod_{ij \in S^1} R_q(g_{ij})$, R_q is a rep. of G

Algebraic higher symmetry $n\mathcal{V}\mathrm{ec}_G$



- The ground state of the bosonic Hamiltonian H is a product state $|0\rangle = \otimes_{ij}|g_{ij} = 1\rangle$.
- The ground state is symmetric, since the action of the symmetry transformations on the ground subspace is proportional to identity $W_q(S^1)|0\rangle = \dim(R_q)|0\rangle$.
- A $(n-1)$-dimensional excitation h on top of the the ground state: changing $g_{ij} = 1$ to $g_{ij} = h$ on a $(d-1)$-dimensional closed subspace = a charge object of the Algebraic $(n-1)$-symmetry.
- Mearsure charge $W_q(S^1)|h\rangle = \mathrm{Tr} R_q(h)|h >$. h and $h' = ghg^{-1}$ carry the same charge.



- The symmetry satisfies $W_{q_1}(S^1)W_{q_2}(S^1) = \sum_{q_3} N^{q_3}_{q_1q_2} W_{q_3}(S^1)$, which is not a group algebra for non-Abelian G.
  The $(n-1)$-dimensional excitations form a local fusion n-catgeory $n\mathcal{V}\mathrm{ec}_G$, which describes the algebraic $(n-1)$-symmetry generated by the Wilson loops $W_q(S^1)$.

Categorical symm. and Algebraic higher symm.

- A **gapless state** is very special, and has a lot of emergent symmetries. The full emergent symmetry may be the **categorical symmetry** Ji-Wen arXiv:1912.13492

- A **categorical symmetry** (= grav. anomaly) is fully characterized by a **topological order** in one higher dimension.
- **Categorical symm.** (= grav. anomaly = bulk topo. order) may completely determines the **minimal gapless state**.

- We can classify all **gapped liquid phases** in systems with a **categorical symmetry**. Such a classification includes
- SETs with **algebraic higher symm.**
- SPTs with **algebraic higher symm.**
- Gauge the **algebraic higher symm.**
- Anomalous **algebraic higher symm.**

Kong, Lan, Wen, Zhang, Zheng; arXiv:2003.08898; arXiv:2005.14178