(Non)-Commutative Field Theories for some Composite Fermi Liquids

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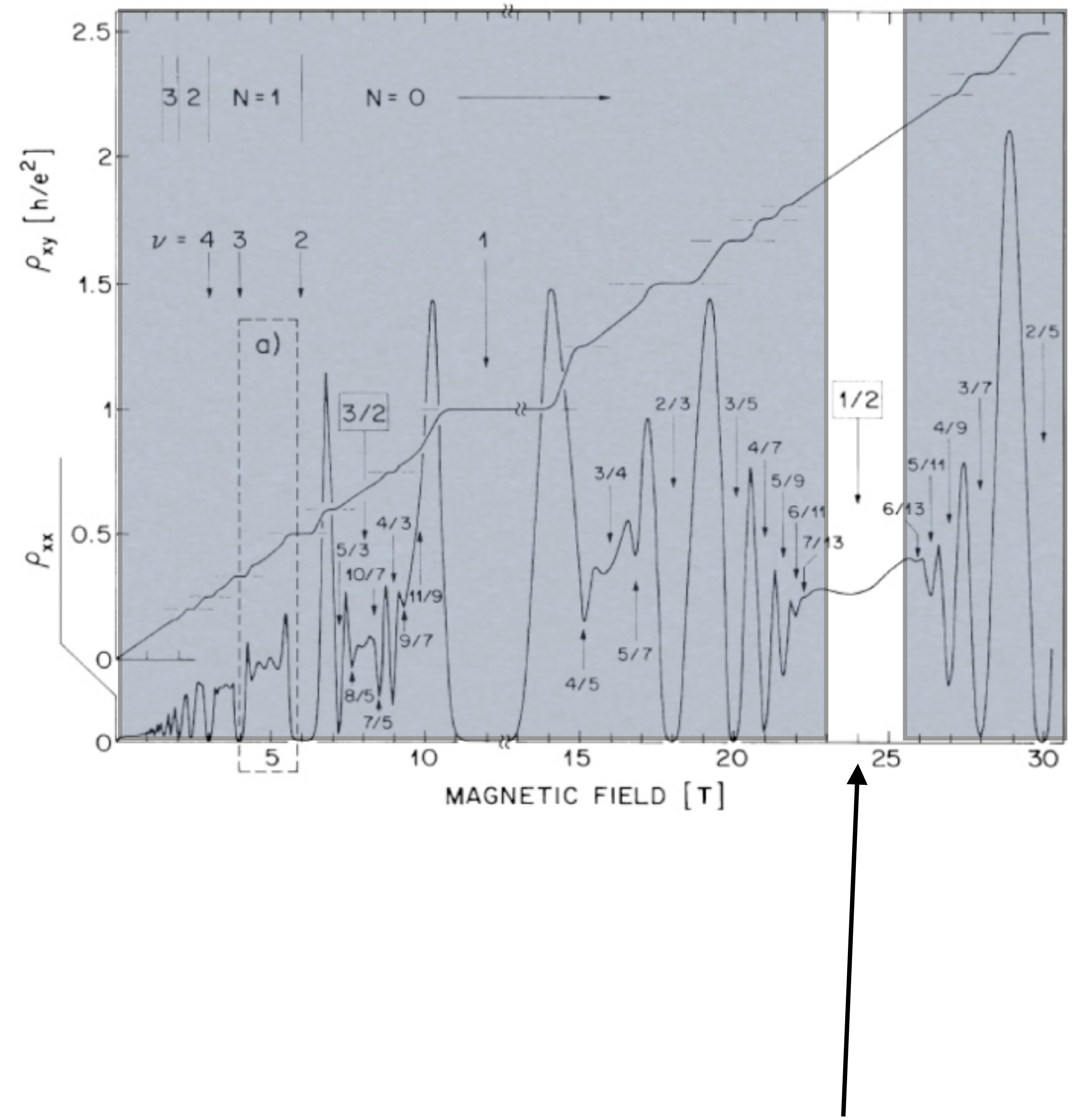
2d electrons in the ``quantum Hall” regime

Filling factor ν = 1,2., 3, ... (IQHE)

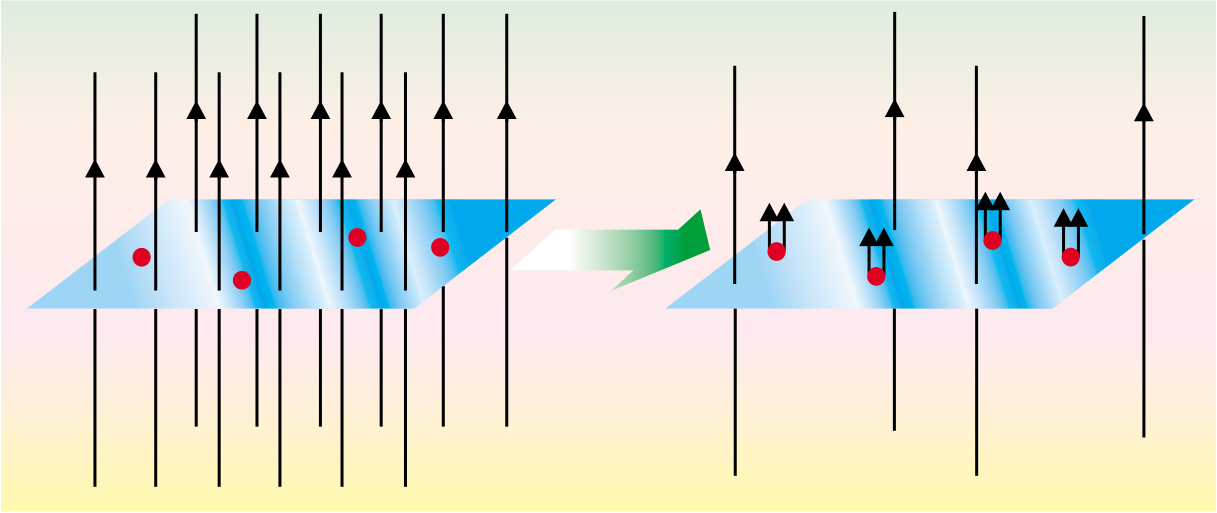
ν = 1/3, 1/5, ... (FQHE)

ν = 1/2, 1/4,…: Metal with $\rho_{xx} \neq 0$, $\rho_{xy} \neq 0$, but $\rho_{xx} \ll \rho_{xy}$. (Composite Fermi Liquids)



Composite fermi liquid theory (Halperin, Lee, Read (HLR) 1993)

Assume (Jain 89) each electron captures two flux quanta to form a new fermion ("Composite fermions")



See reduced effective field
B* = B - (2h/e)ρ

At ν = 1/2, B* = 0

=> form Fermi surface of composite fermions

Effective theory:

$$\mathcal{L} = \overline{\psi}_{CF}\left(i\partial_t - a_0 - iA_0^{ext} + \frac{(\vec{\nabla} - i(\vec{a} + \vec{A}))^2}{2m}\right)\psi_{CF} + \frac{1}{8\pi}a_\mu \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda \quad (1)$$

(Short-hand for better version with properly quantized Chern-Simons terms)

Other composite fermi liquids

1. Electrons at ν = 1/4,…..

2. Bosons at filling ν = 1 (numerics: ground state is a version gapped by pairing but the metallic state is still worth studying)

In these cases too, the HLR construction admits a composite fermi liquid ground state

Effective theory: Fermi surface + U(1) gauge field with Chern-Simons coupling

The many successes of HLR

Explanation/succesfull prediction of many aspects of the observed metallic state

Parent state of nearby Jain sequence of prominent quantum Hall states

Parent state of paired non-abelian quantum Hall state at same filling

Successful backing from numerics

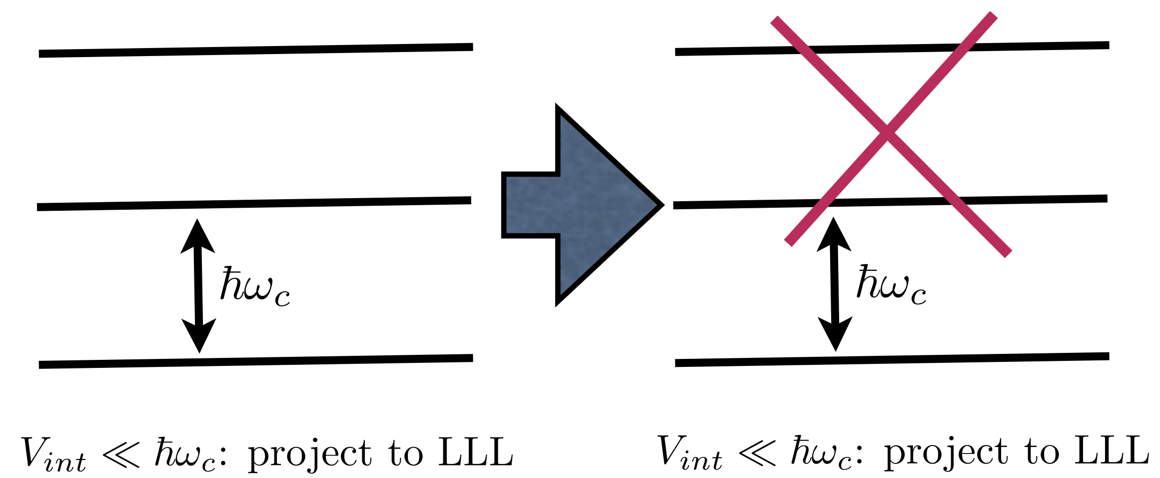
Despite these successes there were some vexing open questions that were extensively thought about in the 1990s without resolution.

Unsatisfactory aspects of the HLR theory-I

Theory should make sense within the Lowest Landau Level (LLL) but HLR not suited to projecting to LLL.

Mean field effective mass = bare electron mass in HLR

LLL limit: take m to zero; what happens??

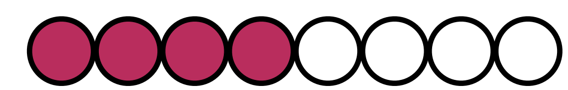


Much discussed in the late 90s (Shankar, Murthy; Read; Halperin, Stern, Simon, van Oppen; D.-H. Lee, Pasquier, Haldane,.......) but dust never settled.

Experience of learning to work within LLL will be useful in other contexts.

Unsatisfactory aspects of the HLR theory-II

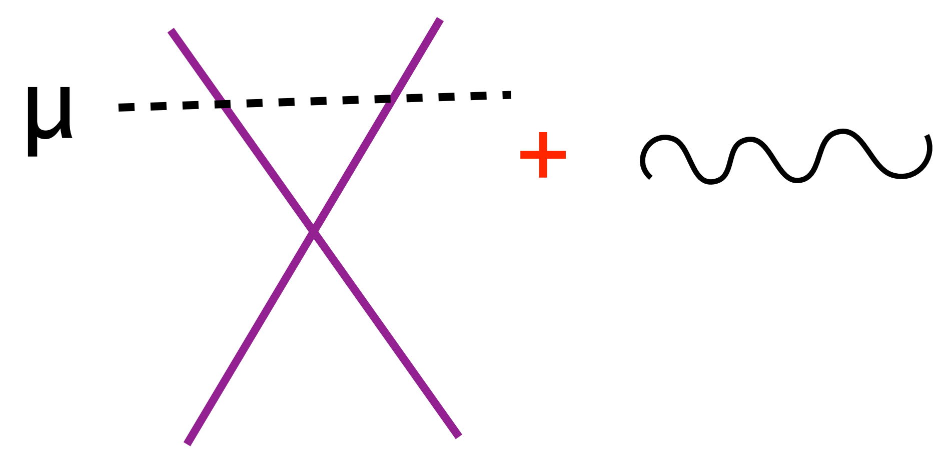
At $\nu = 1/2$ LLL theory has an extra particle-hole symmetry that HLR is blind to.



Issue identified in the 90s (Grotov, Gan, Lee, Kivelson, 96; Lee 98) but a possible resolution only in last few years

``Dirac composite fermions"

Son + many others (2015 - present)



Dirac fermions at finite density + U(1) gauge fields

This progress sidestepped issue of LLL projection;

I will not discuss p/h symmetry today.

The basic problem

Particles in lowest Landau level with 2-body repulsive interactions.

$|m\rangle$: Some basis for single particle states in LLL

Many body states spanned by $|m_1, m_2, m_3, ..., m_N\rangle$ (anti)symmetrized for (fermions) bosons.

Hamiltonian has only interaction energy

$$\mathcal{H} = \frac{1}{2}\int \frac{d^2\mathbf{q}}{(2\pi)^2} U(\mathbf{q})\rho_L(\mathbf{q})\rho_L(-\mathbf{q})$$

Projected "density" operators $\rho_L(\mathbf{q})$ satisfy the "GMP algebra" (a.k.a W_∞ algebra)

$$[\rho_L(\mathbf{q}), \rho_L(\mathbf{q}')] = 2i\sin\left(\frac{(\mathbf{q}\times\mathbf{q}')l_B^2}{2}\right)\rho_L(\mathbf{q}+\mathbf{q}')$$

$U(\mathbf{q}) = e^{-\frac{q^2 l_B^2}{2}} U_0(\mathbf{q})$ with U_0 the Fourier transform of the microscopic 2-body interaction.

Composite Fermi liquid of bosons at $\nu = 1$: Important progress in the 1990s

Pasquier-Haldane (1998): Represent density operator satisfying GMP algebra in terms of fermionic partons (= LLL version of composite fermions)

Highly redundant description; impose constraints to recover physical Hilbert space. Constraint operators themselves satisfy GMP algebra (with opposite sign).

Read(1998): ``W_∞ gauge structure" broken to U(1) by mean field composite fermi liquid solution.

Low energy theory: Fermi surface + U(1) gauge field

Fluctuation effects treated diagrammatically in a conserving approximation;

Physically sensible final results.

Not much further development since…..

Open questions about Pasquier-Haldane-Read

1. What is the low energy effective field theory?

2. How related to HLR?

3. How to treat competition with paired bosonic Pfaffian at same filling? Jain states at proximate filling?

4. Can these methods be generalized to other problems?

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A suggested effective field theory

Composite fermion as a fermionic vortex

$$\mathcal{L}_{vcfl} = \bar{\psi}_v(\partial_\tau + ia_0)\psi_v + \frac{1}{2m^*}|\,(\partial_i + ia_i)\,\psi_v|^2 - \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda - \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$$

Suggested in Read's original 1998 paper but not explicitly derived.

No Chern-Simons term in action; contrast with usual HLR (with renormalized parameters).

$$\mathcal{L}_{HLR} = \bar{\psi}(\partial_\tau + i(a_0 + A_0))\psi_v + \frac{1}{2m^*}|\,(\partial_i + i(a_i + A_i))\,\psi_v|^2 - \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$$

Do they describe the same IR physics, or are they distinct theories?

Which of these is actually obtained in the Pasquier-Haldane-Read microscopic theory?

See also: Alicea, Hermele, Motrunich, Fisher(2005) - flux attachment to fermionize vortices (not in LLL)
Wang, TS (2016) - standard HLR + Fermi surface Berry phase

Our answers

1. This bosonic CFL is described by a ***non-commutative field theory*** of fermions + U(1) gauge field with no Chern-Simons term

2. Approximate map(*) to commutative field theory for long wavelength, low amplitude gauge fluctuations: HLR (+ subheading corrections) with parameters determined by interactions

(*) Using Seiberg, Witten (1999)

Remarks on non-commutative field theory

Field theories defined in a space-time where spatial coordinates do not commute.

Long history in physics/math literature; much studied in high energy physics 1998-2003

Of course the best example of such a space is the Landau Level - so perhaps not surprising that LLL effective theories are non-commutative.

Incompressible states: Proposals for a non-commutative Chern-Simons description (Susskind 2001, Polychronakos 2002,…….)

Connection to microscopics? Added value over standard commutative TQFT description?

Here I will focus on the metallic state of bosons at $\nu = 1$.

Pasquier-Haldane-Read parton construction

Represent states in many body boson Hilbert space by

$$|m_1, ..., m_N\rangle = \epsilon^{n_1 n_2 ..... n_N} c^\dagger_{n_1 m_1} c^\dagger_{n_2 m_2} ..... c^\dagger_{n_N m_N} |0\rangle$$

Symmetric

Anti-symmetric Tensor

c_{mn}: usual fermion anticommutation relations; destroy fermionic partons.

Physical states: singlets under $SU(N)$ rotations on the right:

$$c_{mn} \rightarrow c_{mn'} U^R_{n'n}$$

Generators $\rho^R_{nn'} = c^\dagger_{nm} c_{mn'}$ of these $SU(N)_R$ satisfy constraint:

$$\rho^R_{nn'} |\psi_{phys}\rangle = \delta_{nn'} |\psi_{phys}\rangle$$

'Left' $SU(N)$ rotations generated by $\rho^L_{mm'} = c^\dagger_{nm'} c_{mn}$ are physical operators.

Take $N \rightarrow \infty$, and $|m\rangle, |n\rangle$ as single particle basis states for a Landau level.

Momentum space formulation

Fourier transform c_{mn} using magnetic translation operator $\tau_{\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{R}}$.

$$c_{mn} = \int \frac{d^2k}{(2\pi)^2} \langle m|\tau_{\mathbf{k}}|n\rangle c_{\mathbf{k}}$$

Usual fermion operator

Guiding center coordinate

Fourier-transformed density operators (= fermion bilinears) satisfy:

$$[\rho^L(\mathbf{q}), \rho^L(\mathbf{q}')] = 2i \sin\left(\frac{(\mathbf{q}\times\mathbf{q}')l_B^2}{2}\right) \rho^L(\mathbf{q}+\mathbf{q}')$$ GMP algebra

$$[\rho^R(\mathbf{q}), \rho^R(\mathbf{q}')] = -2i \sin\left(\frac{(\mathbf{q}\times\mathbf{q}')l_B^2}{2}\right) \rho^R(\mathbf{q}+\mathbf{q}')$$ anti-GMP algebra

$$[\rho^L(\mathbf{q}), \rho^R(\mathbf{q}')] = 0$$

The Hamiltonian $\mathcal{H} = \frac{1}{2}\int \frac{d^2\mathbf{q}}{(2\pi)^2} U(\mathbf{q})\rho_L(\mathbf{q})\rho_L(-\mathbf{q})$

Constraint $\rho^R(\mathbf{q})|phys\rangle = 0$ for all non-zero $\mathbf{q}$.

Hartree-Fock solution

$\mathcal{H}$ is quartic in fermions - first study within Hartree-Fock.

A simple (global) symmetry preserving state:

$$\langle c^\dagger_{\mathbf{k}} c_{\mathbf{k}'} \rangle = n_{\mathbf{k}} \delta^{(2)}(\mathbf{k} - \mathbf{k}')$$

Fermions acquire a dispersion (*) and form a Fermi sea with area set by their density.

Mean field composite fermi liquid

(*) Approximate as parabolic - what matters is near Femi surface anyway.

Mean field action:

$$\mathcal{S}_{HF} = \int d\tau \frac{d^2\mathbf{k}}{(2\pi)^2} \;\; \bar{c}_{\mathbf{k}} \frac{dc_{\mathbf{k}}}{d\tau} - \left(\frac{\mathbf{k}^2}{2m^*}\right) c^\dagger_{\mathbf{k}} c_{\mathbf{k}}$$

Beyond mean field

Read (1998): Diagrammatic conserving approximation but here we seek an effective Lagrangian.

Mean field breaks infinite gauge symmetry generated by ρ^R - the breaking is weak as $\mathbf{q} \to 0$.

=> Include small-q gauge fluctuations on top of mean field

Difficulty: In k-space, mean field state is simple but ρ_R transformations mix states at different momenta.

In m, n (Landau orbital) basis, gauge transformations look simple but mean field state is complicated.

Convenient formulation:
Fields in non-commutative space and time

Define abstract fields as a function of the non-commuting guiding center coordinate

$$c(\mathbf{R},\tau) = \int \frac{d^2\mathbf{k}}{(2\pi)^{\frac{1}{2}}} e^{i\mathbf{k}\cdot\mathbf{R}} c_{\mathbf{k},\tau}$$

The Fourier transform definition helps deduce many of their properties.

For instance, products of such fields are given by the ``star product" which is associative but not commutative:

$$f(\mathbf{R}) * g(\mathbf{R}) = e^{i\frac{\theta_{ij}}{2}\partial_{R_i}\partial_{R'_j}} f(\mathbf{R})g(\mathbf{R}')\Big|_{\mathbf{R}'=\mathbf{R}}$$

Here $[R_i, R_j] = i\theta_{ij} = i\Theta\epsilon_{ij}$.
$\Theta = -l_B^2 =$ "non-commutativity" parameter.

Can define derivatives, and integrals

How does this help?

Simple action of transformations generated by $\rho_{L,R}$:

$$c(\mathbf{R},\tau) \to U^L(\mathbf{R},\tau) * c(\mathbf{R},\tau) * U^R(\mathbf{R},\tau)$$

Here take $U^{L,R} = e^{i\theta_{L,R}(\mathbf{R},\tau)}$ with exponential defined through power series with all products being star products.

Mean field action also looks simple:

$$\mathcal{S}_{HF} = \int d\tau d^2\mathbf{R} \quad \left(\overline{c}(\mathbf{R},\tau) * \frac{dc_{\mathbf{R}}}{d\tau} + \frac{1}{2m} \nabla \overline{c}(\mathbf{R},\tau) * \nabla c(\mathbf{R},\tau) \right)$$

Now require that action be invariant under long wavelength `right' gauge transformations:
Replace all derivatives by covariant derivatives

Non-commutative action for composite fermi liquid

$$\mathcal{S} = \int d^2\mathbf{R}d\tau \;\; \bar{c} * D_0 c + i a_0 \underline{\rho} + \frac{1}{2m^*}\overline{D_i c} D_i c$$

The covariant derivatives

$$D_\mu c = \partial_\mu c - i c * a_\mu - i A_\mu * c$$

a_μ: dynamical non-commutative $U(1)$ gauge field coupling to 'right' currents.

A_μ: background non-commutative $U(1)$ gauge field coupling to 'left' currents

Fermions at finite density, form a Fermi surface; gauge fields vary slowly on scale l_B.

Gauge invariance:

$$\begin{aligned} c &\rightarrow c + i c * \theta_R + i\theta_L * c \\ a_\mu &\rightarrow a_\mu + \partial_\mu \theta_R + i(a_\mu * \theta_R - \theta_R * a_\mu) \\ A_\mu &\rightarrow A_\mu + \partial_\mu \theta_L + i(\theta_L * A_\mu - A_\mu * \theta_L) \end{aligned}$$

Comments

1. Theory has Fermi surface + dynamical U(1) gauge field without any Chern-Simons term but is formulated in non-commutative space-time.

2. Hartree-Fock theory gives estimate of bare effective mass m* in terms of interaction strength.

How to compare with previous proposals?

Previous proposed field theories (HLR, fermionic vortex liquid) are all ordinary commutative field theories.

However, the non-commutativity occurs at scale of magnetic length, we can hope to approximate long wavelength gauge fluctuations in terms of a `coarge-grained' commutative theory.

Key tool: The Seiberg-Witten map

Seiberg, Witten (1999): Map pure non-commutative gauge theory with gauge fields a_μ to a commutative gauge theory with gauge fields $\hat{a}_\mu$ in a systematic expansion in powers of Θ.

Expansion is local (coefficients only involve fields and derivatives at same space-time point).

Gauge transformation parameters are mapped as functions of the gauge field configurations themselves.

Here we will need a small generalization that includes the fermion fields.

Pure $U(1)$ gauge theory: Exact an non-perturbative Seiberg-Witten map exists (Liu, Michelson, Okawa, Ooguri 01, Mukhi, Suryanarayana 01).

Appealing physical interpretation: Relation to map between Lagrangian and Eulerian descriptions of a fluid (Jackiw, Pi, Polychronakos, 02; also Susskind 01)

Seiberg-Witten map to linear order

$$
\begin{aligned}
A(\hat{A}) &= \hat{A} + \Delta A(\hat{A}) & a(\hat{a}) &= \hat{a} + \Delta a(\hat{a}) \\
\theta_L(\hat{\theta}_L, \hat{A}) &= \hat{\theta}_L + \Delta\theta_L(\hat{\theta}_L, \hat{A}) & \theta_R(\hat{\theta}_R, \hat{a}) &= \hat{\theta}_R + \Delta\theta_R(\hat{\theta}_R, \hat{a}) \\
c(\psi, \hat{A}, \hat{a}) &= \psi + \Delta\psi(\psi, \hat{A}, \hat{a})
\end{aligned}
\tag{1}
$$

Here ΔA, Δa, $\Delta\theta_R$, $\Delta\theta_L$, and Δc are all of $o(\Theta)$.

The map can be found explicitly by requiring that the hatted fields satisfy ordinary U(1) gauge invariance for both dynamical and background gauge fields.

Strategy: Plug in the map to get an effective commutative gauge theory

Emergence of HLR

Effective commutative Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{HLR} + \mathcal{L}_{corr} \\
\mathcal{L}_{HLR} &= \bar{\psi}\partial_0\psi - i(\hat{a}_0 + \hat{A}_0)\bar{\psi}\psi + i\hat{a}_0\underline{\rho} + \frac{1}{2m^*}\left|\Big(\partial_i - i(\hat{a}_i + \hat{A}_i)\Big)\psi\right|^2 - i\frac{1}{4\pi}\epsilon^{\alpha\beta\gamma}\hat{a}_\alpha\partial_\beta\hat{a}_\gamma \\
\mathcal{L}_{corr} &= -\frac{\Theta}{2}\epsilon^{\alpha\beta}\left((\hat{f}_{0\beta} - \hat{F}_{0\beta})\partial_\alpha(\bar{\psi}\psi) - \partial_\alpha(\hat{a}_\beta - \hat{A}_\beta)(\bar{\psi}D_0\psi - \frac{1}{2m^*}|\hat{D}_i\psi|^2)\right)
\end{aligned}
$$

First term is just the standard HLR theory!
(but with correct effective mass)

Second term is a correction that is small for long wavelength, low amplitude gauge fluctuations

Comments

1. The commutative theory has a Chern-Simons term (as expected in HLR) which is absent in the non-commutative theory.

This is traced to the non-trivial form factors in the density operator in the non-commutative theory (roughly correspond to a non-trivial Berry phase)

2. Can use the commutative effective theory to, say, study Jain states near $\nu = 1$ (technically a bit hard within the microscopic Pasquier-Haldane-Read approach) The correction terms may need to be included for some purposes.

3. We do not find the fermionic vortex field theory.

Conclusions/outlook

Other composite fermi liquids?

Some simple generalizations easy (eg spinfull bosons at total filling 1)

Fermions at 1/2-filling?

In all cases it is natural that a LLL theory is non-commutative .

Other generalizations possible - eg mean field theory of
multicomponent quantum-Hall like systems at
total integer filling.