Many-Body Level Statistics

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Outline

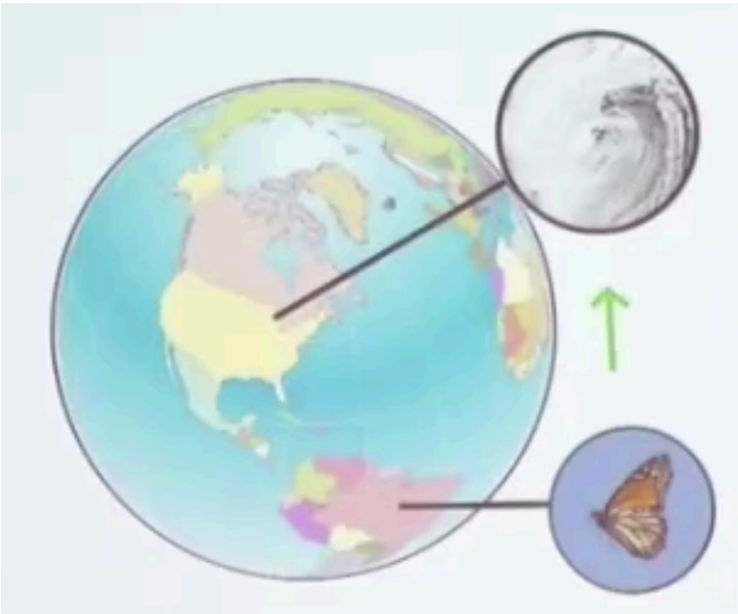
- Introduction
  - Classical ergodic hierarchies
  - Quantum chaos: 3 “definitions”

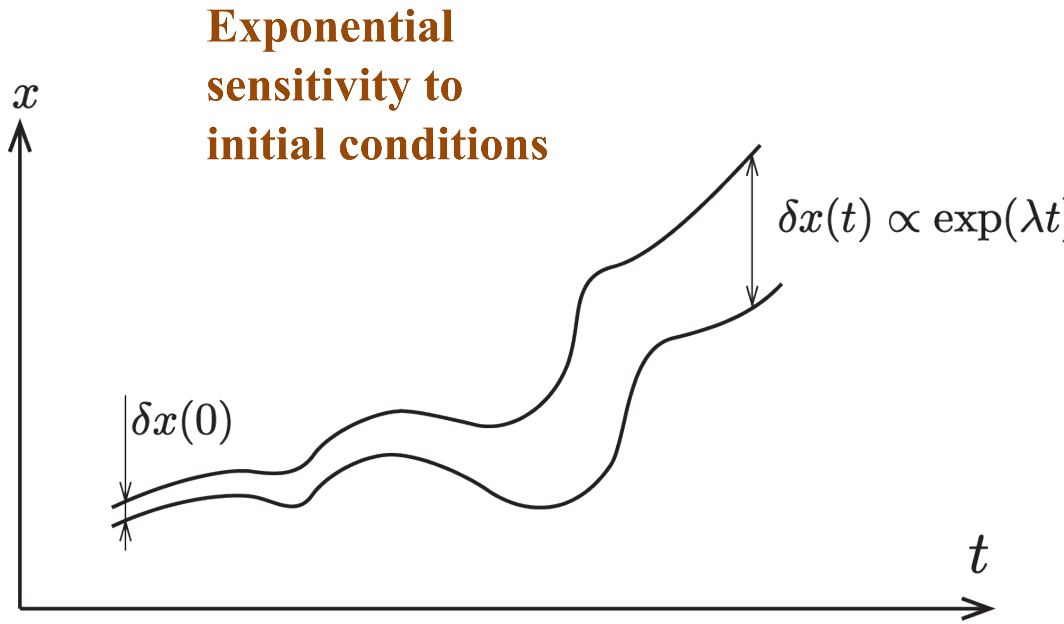
- RMT & Wigner-Dyson statistics
  - Linear ramp in the spectral form factor
  - $RMT = SYK2 \sim \lim_{q \to 0} Disordered\ metal$

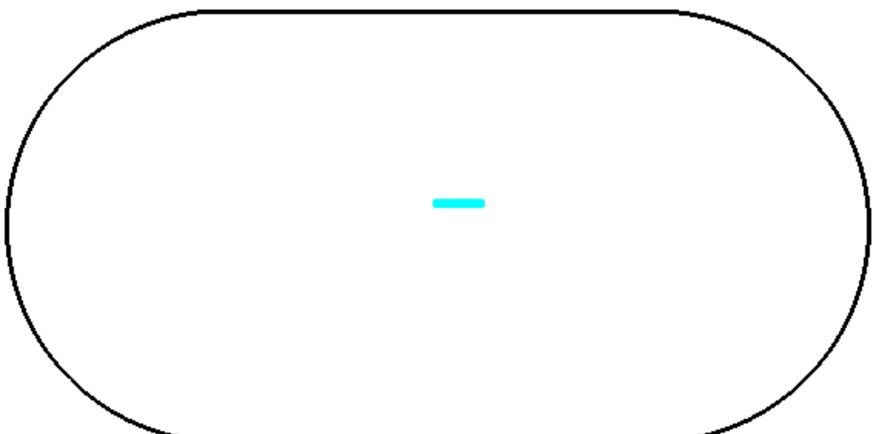
- Many-body level statistics of single-particle chaos
  - The slope, the exp ramp, and the plateau
  - Residual level repulsion & islands of level attraction

- Sigma-model approach to many-body quantum chaos

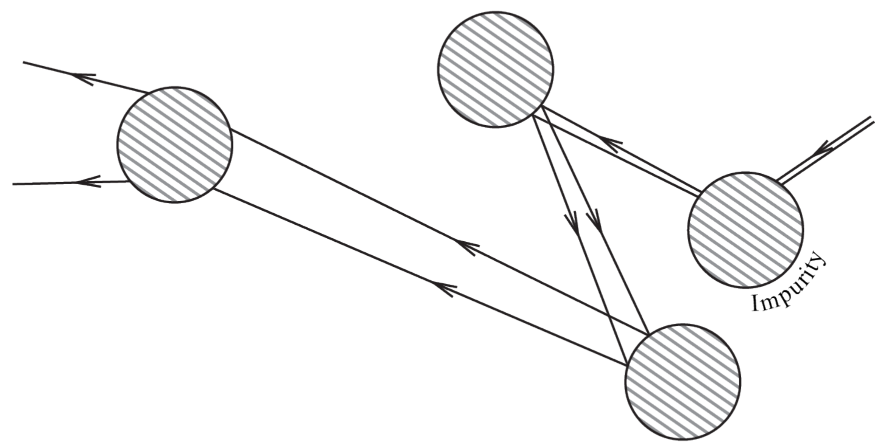
Classical "butterfly effect"





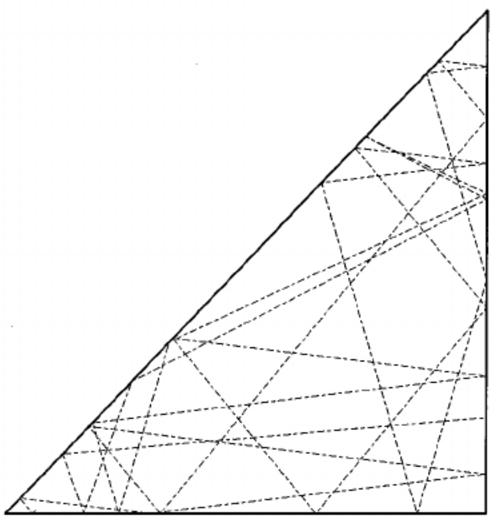


Bunimovich stadium billiard

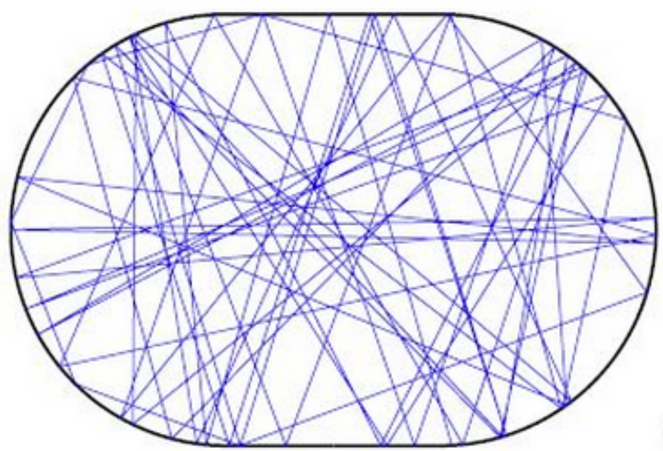


Disordered medium (e.g., a metal)

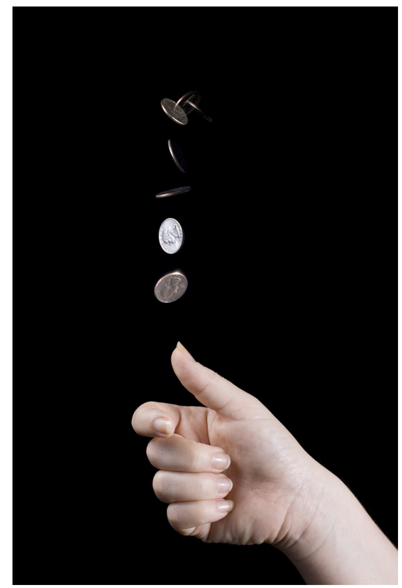
Classical ergodic hierarchy



Irrational triangular billiard
(ergodic, mixing, but not chaotic)



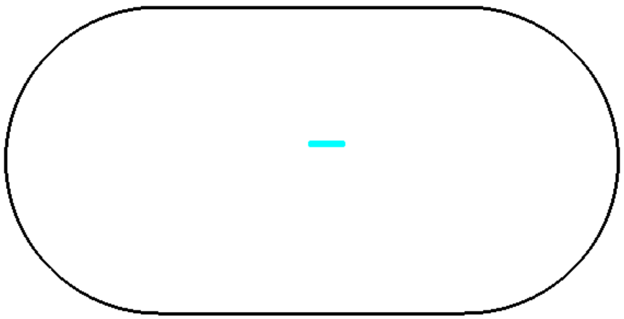
Bunimovich stadium
(ergodic & chaotic)

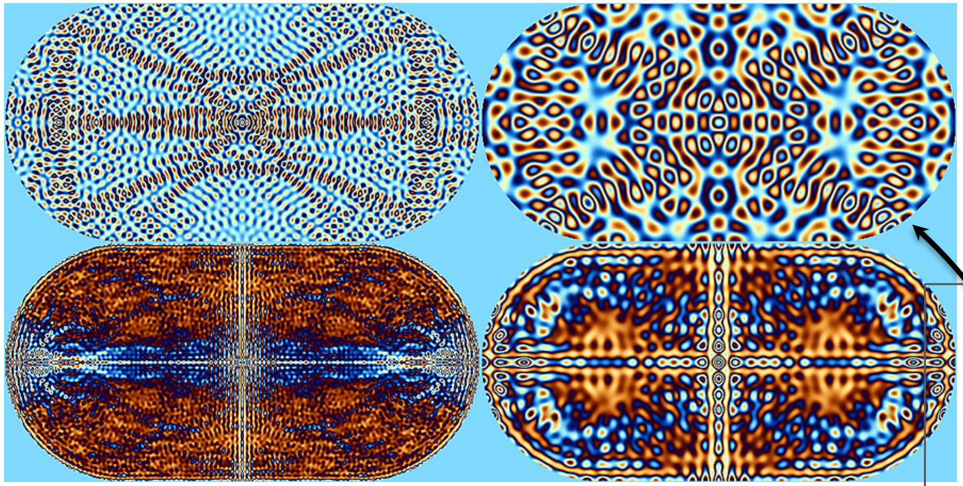


Bernoulli dynamical process
(~coin toss)

Three definitions of "quantum chaos"

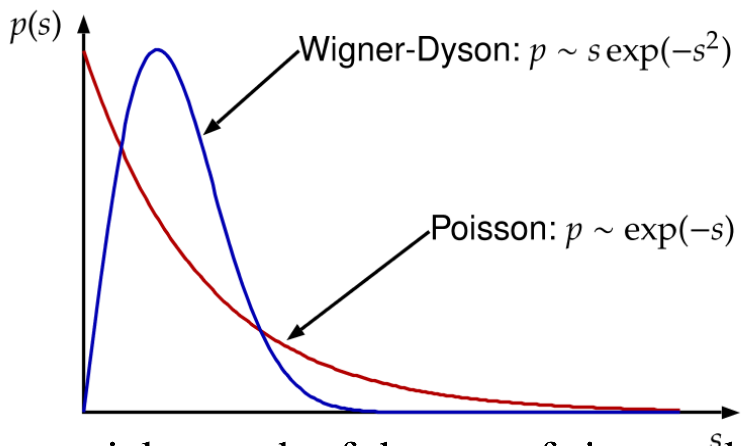
- Quantize a classically-chaotic model. Call whatever you get a "quantum chaotic system."





Bohigas-Giannoni-Schmit (BGS) conjecture

- Look at the energy spectrum of a quantum model (which may or may not have a well-defined classical limit) and compare its energy level statistics with the universal Wigner-Dyson distribution, predicted by random matrix theory.



- "New" diagnostic: exponential growth of the out-of-time-ordered correlator (Larkin & Ovchinnikov, Stanford, Shenker, Kitaev, Maldacena)

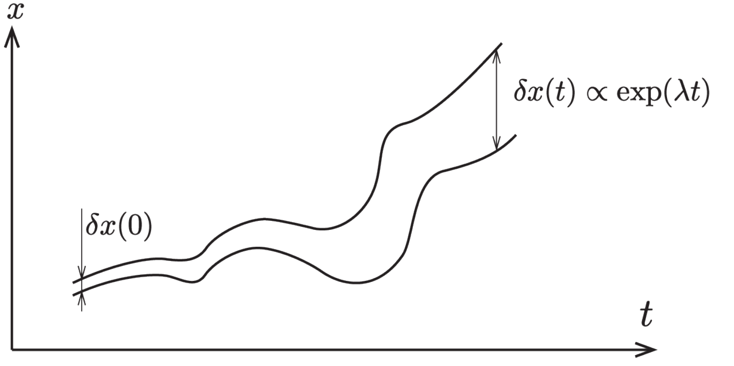
Out-of-time-ordered correlator (OTOC)

- Consider quantum mechanical operators, $\hat{x}(t)$ and $\hat{p}(0) = -i\hbar \frac{\partial}{\partial x(0)}$ and define the correlator (OTOC)

Anatoly Larkin

Yurii Ovchinnikov

$$C_{xp}(t) = -\langle [\hat{x}(t), \hat{p}(0)]^2 \rangle = \hbar^2 \left\langle \left(\frac{\partial \hat{x}(t)}{\partial x(0)} \right)^2 \right\rangle.$$



- If the correlator grows exponentially, we'll call the system quantum chaotic and the corresponding exponent – the Lyapunov exponent.

$$C(t) \propto e^{2\lambda_{max} t}$$

- Such correlators can be defined for many-body systems.
- Kitaev, Maldacena, Stanford et al. conjectured a bound on many-body chaos.

Recent interest in OTOCs is partly due to connections to gravity

- Kitaev, Maldacena, Stanford et al. considered (symmetrized) OTOC functions

$$F(t) = \frac{1}{Z}\mathrm{tr}\left\{e^{-\frac{\beta H}{4}}B(0)e^{-\frac{\beta H}{4}}A(t)e^{-\frac{\beta H}{4}}B(0)e^{-\frac{\beta H}{4}}A(t)\right\} \sim e^{\tilde{\lambda}t}$$

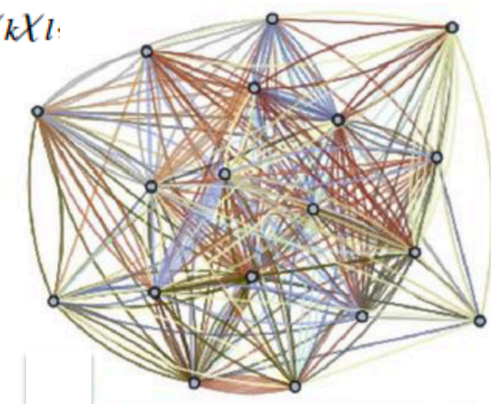
- Using assumptions about exp growth, *certain inequality involving the TOC and OTOC*, & analytical properties in z=t+iβ, a theorem has been proven:

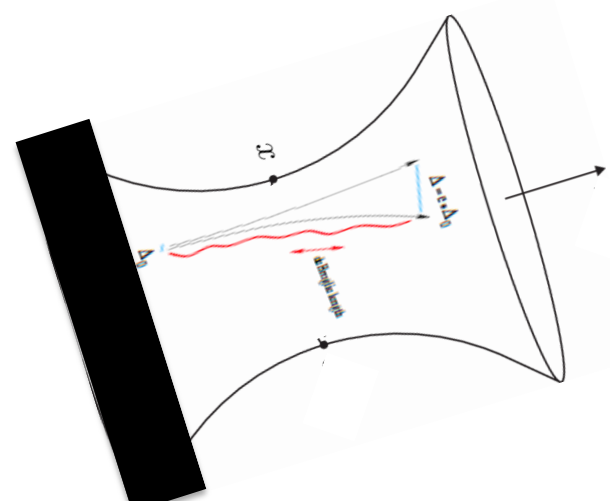
$$\tilde{\lambda} \leq \frac{2\pi k_B T}{\hbar}$$

- Remarkably, the bound saturates for certain strongly-correlated models (e.g. SYK) with interesting gravity-like dual descriptions.

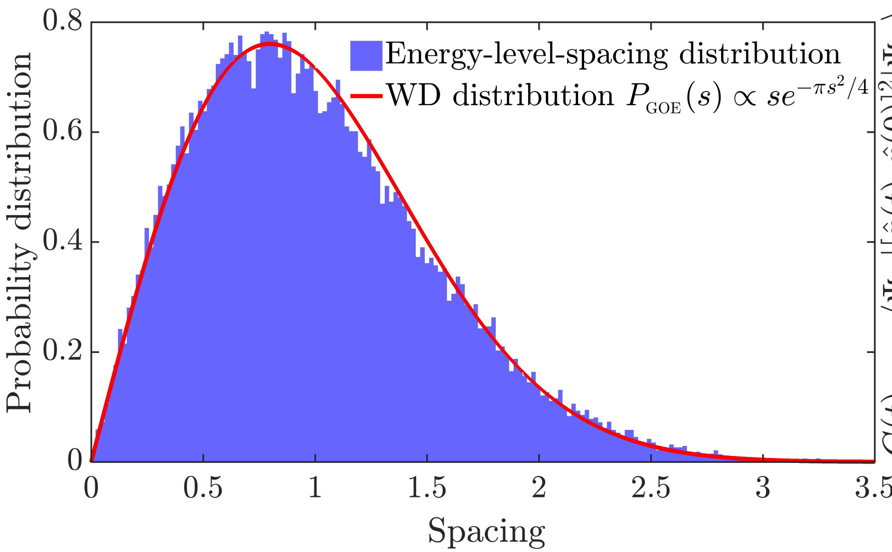
$$\hat{H} = \sum_{ijkl}^{N} J_{ijkl}\chi_i\chi_j\chi_k\chi_l$$

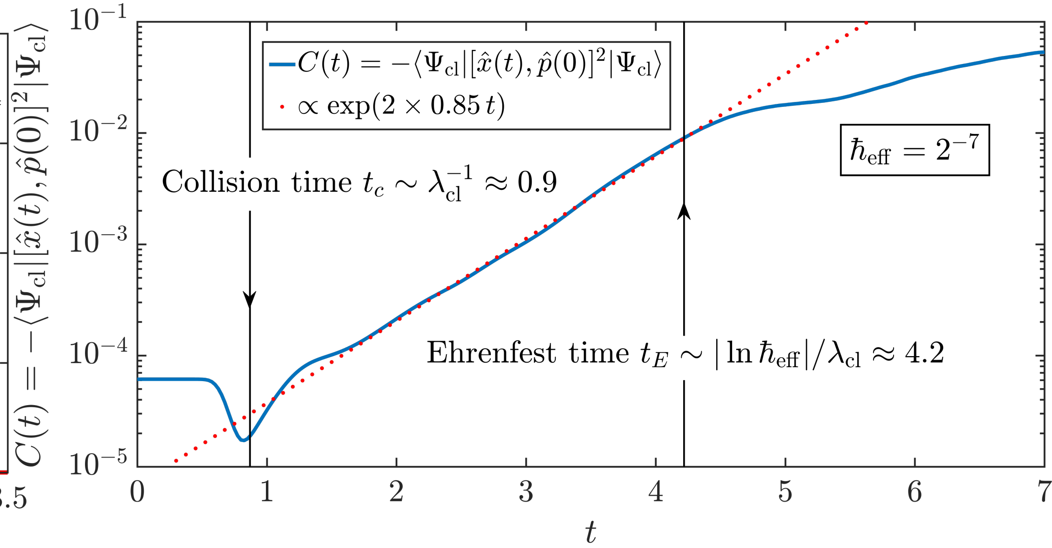
SYK

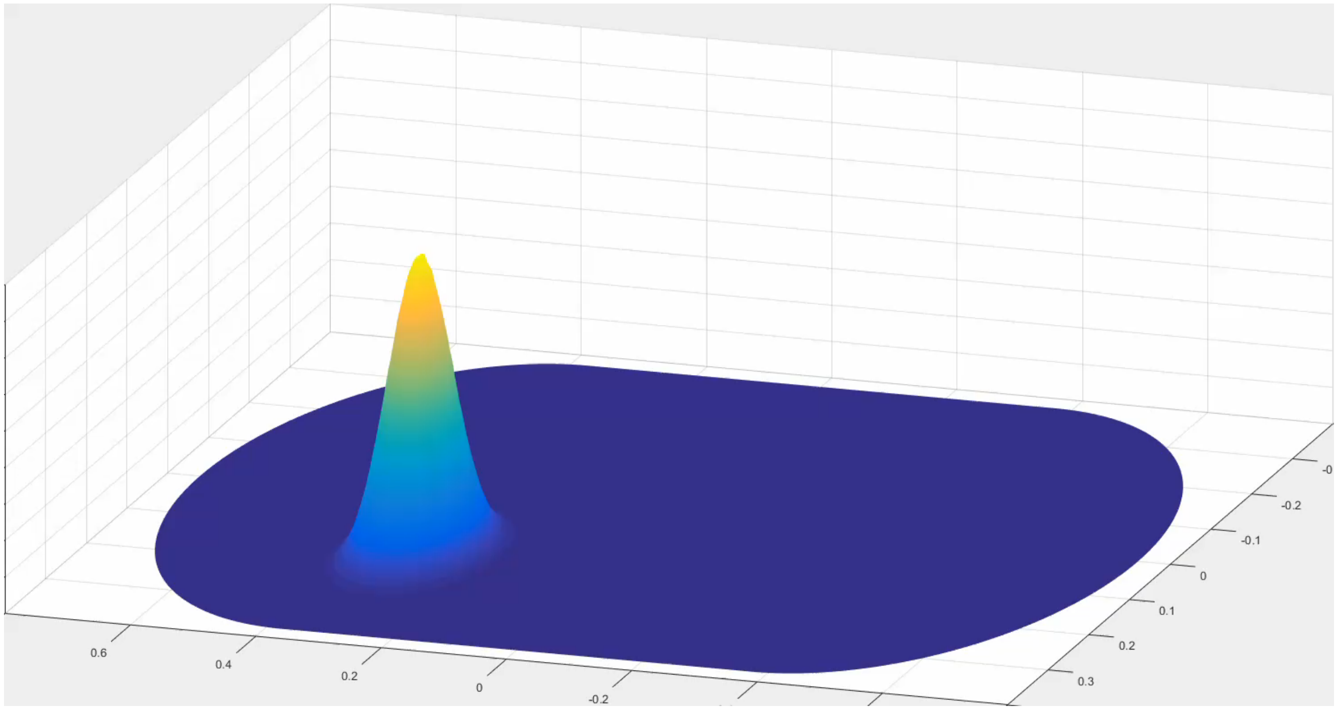




Consistency of definitions: Bunimovich billiard





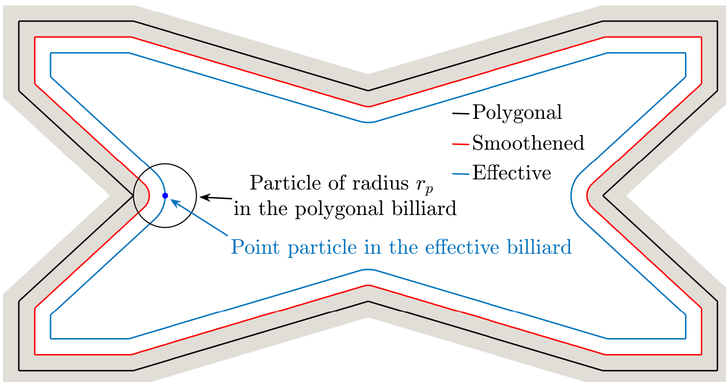


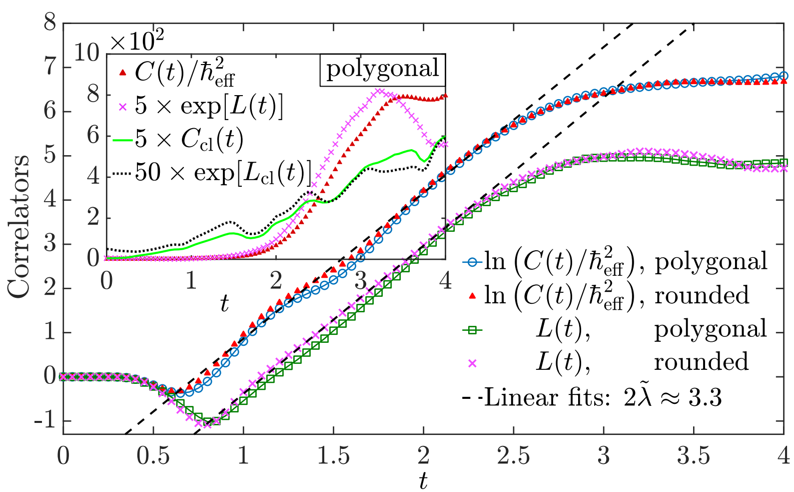
OTOC is not generally a good diagnostic of chaos

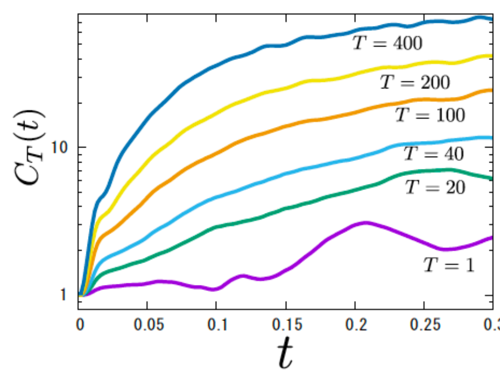
Early–Time Exponential Instabilities in Non–Chaotic Quantum Systems

Efim B. Rozenbaum, Leonid A. Bunimovich, Victor Galitski

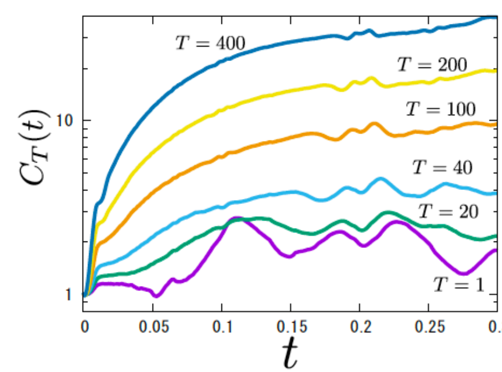
To appear in *Phys. Rev. Lett.*







(a) Stadium billiard ($a/R = 1$)



(b) Circle billiard ($a/R = 0$)

- Some models show "fake" exponential instabilities in OTOC without underling chaos.
- Also, a "wrong" choice of initial conditions in a chaotic model may yield no exponential OTOC. Furthermore, many chaotic/non-integrable many-body models (e.g., spin-1/2 chains) do not have classical states at all, and their OTOC would never show exp growth.

A less "biased" approach to defining & describing "quantum chaos" is through level statistics.

Random matrix ensemble (GUE for simplicity)

- Gaussian ensemble of $N \times N$ Hermitian random matrices, H

$$P(H) = e^{-\frac{N}{2}\mathrm{Tr}\, H^2} dH$$

- Haar measure: $dH = \prod_k dh_{kk} \prod_{i<j} d^2 h_{ij}$

- Diagonalization by unitary rotation $U(\mathbf{v})$ ($\mathbf{v}$ is $N^2 - N$-dimensional)

$$H = U(\mathbf{v})\mathrm{diag}\,(\varepsilon_1, \ldots, \varepsilon_N)\, U^\dagger(\mathbf{v})$$

- $\mathrm{Tr}\, H^2 = \sum_j \varepsilon_i^2$ and Haar measure

$$dH = J(\boldsymbol{\varepsilon}, \mathbf{v}) d^{(N^2-N)} v \prod_k^N d\varepsilon_k$$

- Level repulsion is easy to see. If $\forall \varepsilon_i = \varepsilon_j$, then $\forall \mathbf{v}$

$$U(\mathbf{v}) \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} U^\dagger(\mathbf{v}) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$$ and therefore $J(\varepsilon_i = \varepsilon_j, \mathbf{v}) \equiv 0$

Wigner-Dyson level statistics

- Scaling $J(\alpha\boldsymbol{\varepsilon},\mathbf{v}) = \alpha^{N^2-N} J(\boldsymbol{\varepsilon},\mathbf{v})$ & symmetry $\varepsilon_i \leftrightarrow \varepsilon_j$ leave one outcome

$$P(\boldsymbol{\varepsilon}) \propto e^{-\frac{N}{2}\sum_k \varepsilon_k^2} \underbrace{\prod_{i<j} (\varepsilon_i - \varepsilon_j)^2}_{\text{level repulsion}}$$

- Vandermonde determinant representation

$$\prod_{i<j} (\varepsilon_i - \varepsilon_j) = \det \|\varepsilon_j^{i-1}\|_{i,j=1,\ldots,N}$$

- We can replace ε_j^{i-1} with *any polynomial* $C_{i-1}(\varepsilon_j)$ (of degree $i-1$).

- Hermite polynomials are a smart choice (c.f., harmonic oscillator w.f.)

$$P(\boldsymbol{\varepsilon}) \propto e^{-\frac{N}{2}\sum\limits_{k=1}^{N} \varepsilon_k^2} \det{}^2 \|H_i(\varepsilon_j)\|_{i,j=1,\ldots,N}$$

Correlation functions

- Full WD distribution has "too much" information. We can integrate some of it out

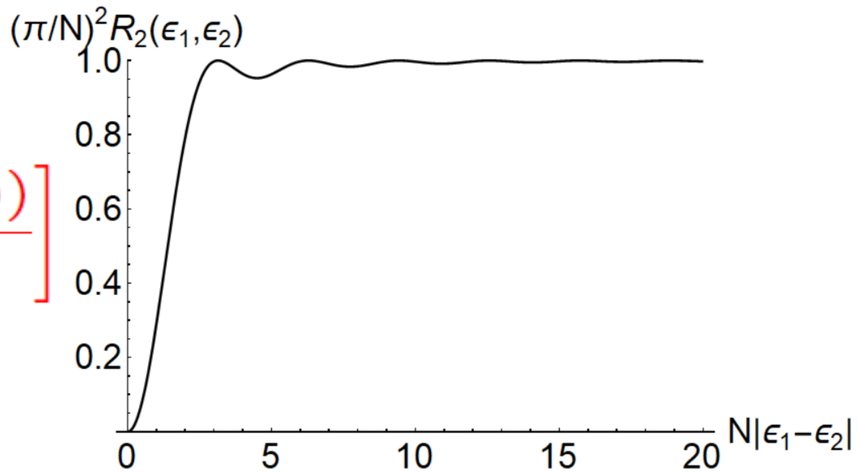
$$R_m(\varepsilon_1, \ldots, \varepsilon_m) = \frac{N!}{(N-m)!} \int d\varepsilon_{m+1} \ldots d\varepsilon_N P(\varepsilon_1, \ldots, \varepsilon_N)$$

- Wigner semicircle law (m=1, i.e., density of states)

$$R_1(\varepsilon) = \frac{N}{2\pi}\sqrt{4-\varepsilon^2}$$

- Two-level correlator (m=2) for $N \to \infty$

$$R_2(\varepsilon_1, \varepsilon_2) \approx \frac{N^2}{\pi^2}\left[1 - \frac{\sin^2(N(\varepsilon_1-\varepsilon_2))}{N^2(\varepsilon_1-\varepsilon_2)^2}\right]$$



- R_2 also appears to be the correlation function of the imaginary part of the non-trivial zeroes of the Riemann zeta function – "Montgomery pair correlation conjecture."

Spectral form factor (SFF) – a probe of level statistics

- To find if a system is “chaotic” in practice means to calculate R_2 or its Fourier T. - SFF

$$K(t) = \langle |Z(it)|^2 \rangle = \sum_{n,m} \left\langle e^{i(E_n - E_m)t} \right\rangle_{\text{ensemble averaging}}$$

- Two obvious regimes: $K(t=0) = N^2$and $K(t \to \infty) = N$ (plateau)

- Roughly, $t \sim 1/\Delta E$, and the plateau reflects the absence of levels with $\Delta E \ll 1/N$

- At small finite t, $t \to 0$ perturbation theory is straightforward. Fall off the initial N^2 cliff:

$$K_{\text{slope}}(t) = N^2 \frac{J_1^2(2t)}{t^2}$$

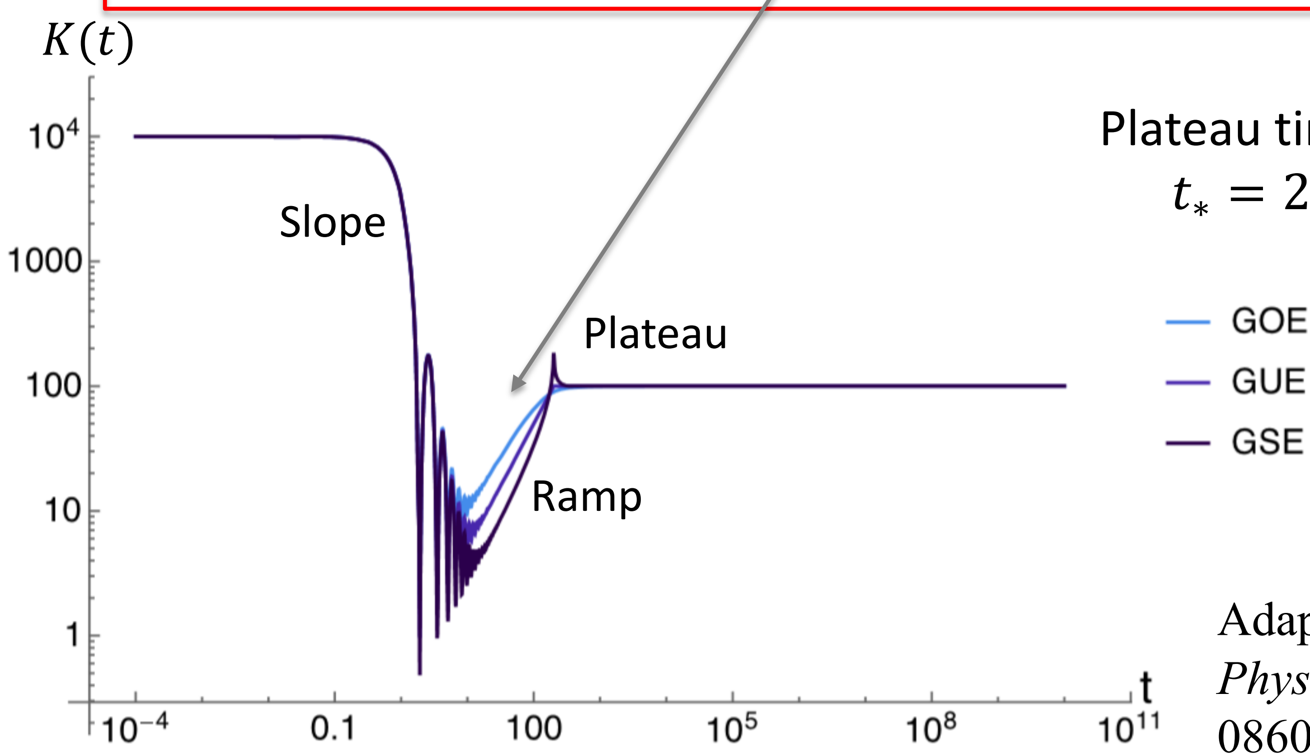
- But there is much structure, including level repulsion, hidden at intermediate “times.”

A linear "ramp" in the SFF – the hallmark of chaos

$$R_2(\varepsilon_1,\varepsilon_2) - R_1(\varepsilon_1)R_1(\varepsilon_2) \propto -\frac{\sin^2[N(\varepsilon_1-\varepsilon_2)]}{N^2(\varepsilon_1-\varepsilon_2)^2}$$

- The second term (level repulsion) gives a linear "ramp" at $t \sim O(N)$

$$-\frac{\sin^2(N(\varepsilon_1-\varepsilon_2))}{N^2(\varepsilon_1-\varepsilon_2)} \xrightarrow{\text{FT}} -\frac{\pi}{N}\left(1-\frac{t}{2N}\right)\theta(2N-t)$$



Plateau time:
$t_* = 2N$

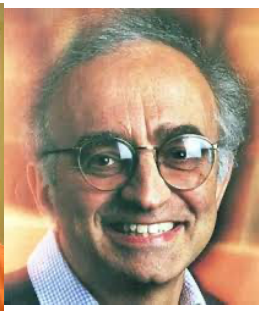
Adapted from J. Liu, *Phys. Rev. D* **98**, 086026 (2018).

$RMT = SYK2 \sim \lim_{q\to 0} Disordered\ metal$

Repulsion of energy levels and conductivity of small metal samples

B. L. Al'tshuler and B. I. Shklovskiĭ

Zh. Eksp. Teor. Fiz. **91**, 220–234 (July 1986)



VOLUME 85, NUMBER 26 PHYSICAL REVIEW LETTERS 25 DECEMBER 2000

Wigner-Dyson Statistics from the Keldysh σ-Model

Alexander Altland[1] and Alex Kamenev[2]

... obtain the well-known result

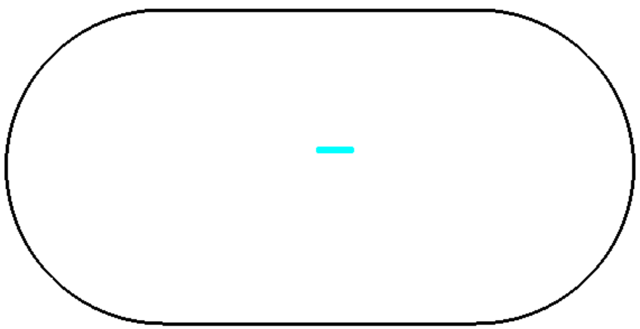
$$R(\omega) = -\frac{1}{2}\,\mathrm{Re}\,\frac{1-\exp(2is)}{s^2} = -\left(\frac{\sin s}{s}\right)^2$$

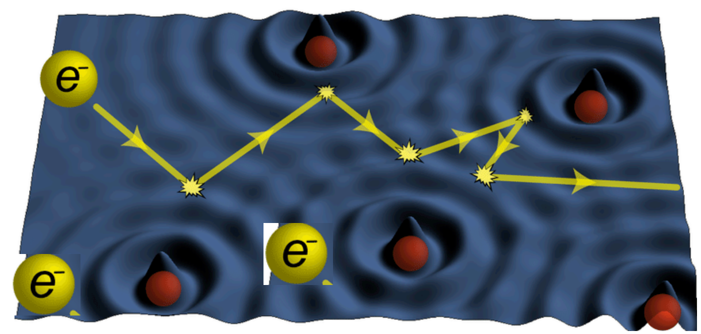
Key insight: The WD result for $R_2(\omega = \varepsilon_1 - \varepsilon_2)$ in the case of a disordered metal hinges on singularities associated with integrating the diffuson modes, $D(\omega, \boldsymbol{q}) = \frac{1}{-i\omega + Dq^2}$

RMT results (we can now call complex SYK2) $H = \sum_{i,j} \overline{\psi_i}\, h_{ij}\psi_j$, can be obtained by taking a 0-dim limit, $q \to 0$, of quantum diffusion and originate from soft modes $\sim 1/\omega$.

But this is NOT many-body quantum chaos

- Both diffusive Altshuler-Shklovski & Altland-Kamenev model and SYK2 are described by quadratic fermionic Hamiltonians. The derived Wigner-Dyson correlators refer to *single-particle* spectrum.
- The both models are integrable and do not thermalize (not ergodic/chaotic)
- These are non-interacting many-body systems ”embedded” in a chaotic medium.





- Many-body energies are obtained by populating the Wigner-Dyson s.p. levels.

$$E_\alpha = \sum_{k=1}^{N} n_k^{(\alpha)} \varepsilon_k$$

- The occupation numbers, $n_k^{(\alpha)} = 0{,}1$ correspond to many-body Fock states, $|\alpha\rangle$. The Hilbert space size (# of levels E_α) is $L = 2^N$, with $N \gg 1$. The structure of the many-body energy landscape is much finer & it should NOT be Wigner-Dyson.

What is many-body statistics of single-particle chaos?

arXiv.org > cond-mat > arXiv:2005.08991

Condensed Matter > Statistical Mechanics

[Submitted on 18 May 2020]

Many-body level statistics of single-particle quantum chaos

Yunxiang Liao, Amit Vikram, Victor Galitski

Model – an ensemble of non-interacting fermion systems with N GUE-distributed single particle levels (complex SYK-2) and $L = 2^N$ many-body levels

$$\hat{H} = \sum_{i,j} \hat{f}_i^{\dagger} (h_{ij} - \mu \delta_{ij}) \hat{f}_j$$

$$P(h) = 2^{N(N-1)/2} \left(\frac{N}{2\pi} \right)^{N^2/2} \exp\left[-\frac{N}{2} \operatorname{Tr}\left(h^2\right) \right]$$

Many-body SFF – series expansion

- $Z(it)$ is easy to evaluate for a non-interacting system, leaving us with a GUE ensemble average

$$K(t) = \left\langle \sum_{\alpha,\beta=1}^{L} \prod_{k=1}^{N} e^{i(n_k^\alpha - n_k^\beta)(\varepsilon_k - \mu)t} \right\rangle$$

$$= 2^N \int d\varepsilon_1 ... d\varepsilon_N P(\varepsilon_1, ..., \varepsilon_N) \prod_{i=1}^{N} \{1 + \cos[(\varepsilon_i - \mu)t]\}$$

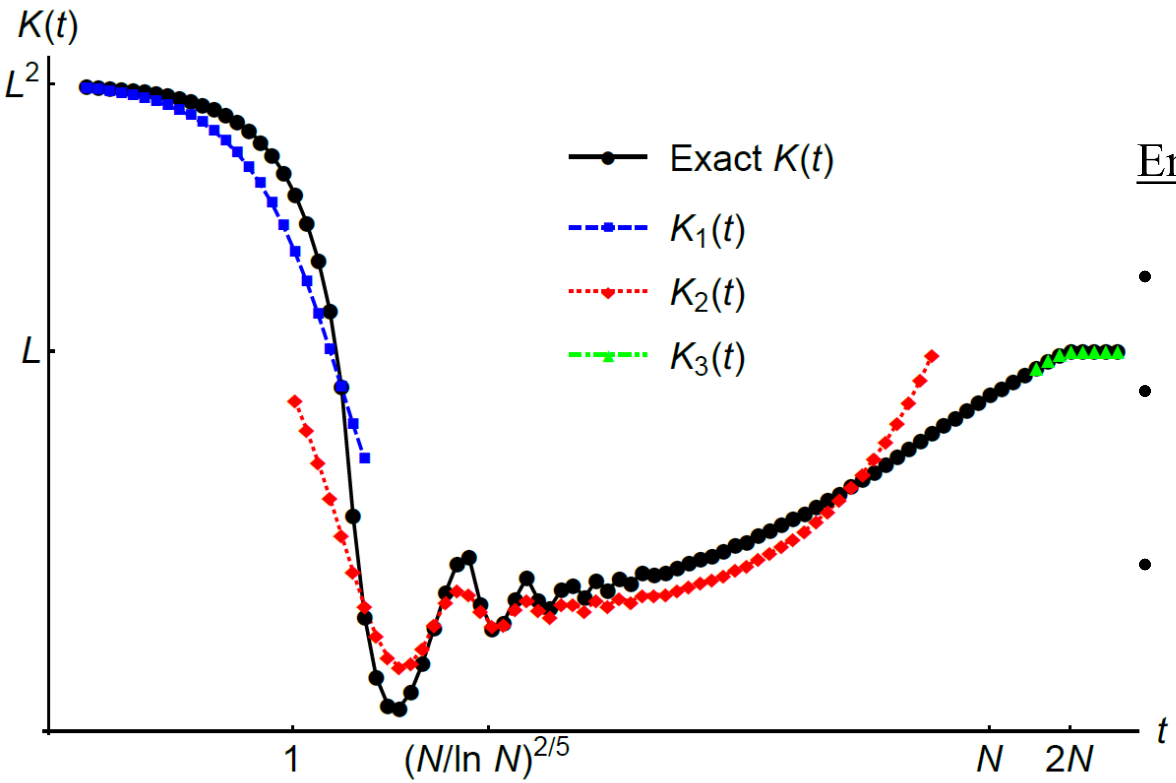
$$= 2^N \left(1 + \sum_{n=1}^{N} \frac{1}{n!} \bar{\mathtt{r}}_n(t)\right)$$

$$\bar{\mathtt{r}}_n(t) = \int \mathrm{d}\varepsilon_1 ... \mathrm{d}\varepsilon_n \mathrm{R}_n(\varepsilon_1, ..., \varepsilon_n) \prod_{i=1}^{n} \cos[(\varepsilon_i - \mu)t]$$

... after a lot of algebra and combinatorics, which I am going to skip, we arrive at…

SFF for non-interacting "quantum chaotic" fermions

$$K(t) \approx \begin{cases} K_1(t) = L^2 \cos^{2N}\left(\frac{\mu t}{2}\right) \exp\left[N\left(\frac{J_1(2t)}{t} - 1\right)\cos(\mu t)\right], & 0 < t \ll O(1), \\ K_2(t) = \left(\frac{N}{8} e^{\gamma_E}\right)^{t/4} \exp\left[N \frac{J_1(2t)}{t} \cos(\mu t)\right], & O(1) \ll t < \eta_1 < O\left(N/\log_2 N\right), \\ K_3(t) = L \exp\left[-\frac{(4N^2 - t^2)^{\frac{3}{2}}}{12\pi N t}\Theta(2N - t)\right], & \sqrt{2}N < t < \infty \end{cases}$$

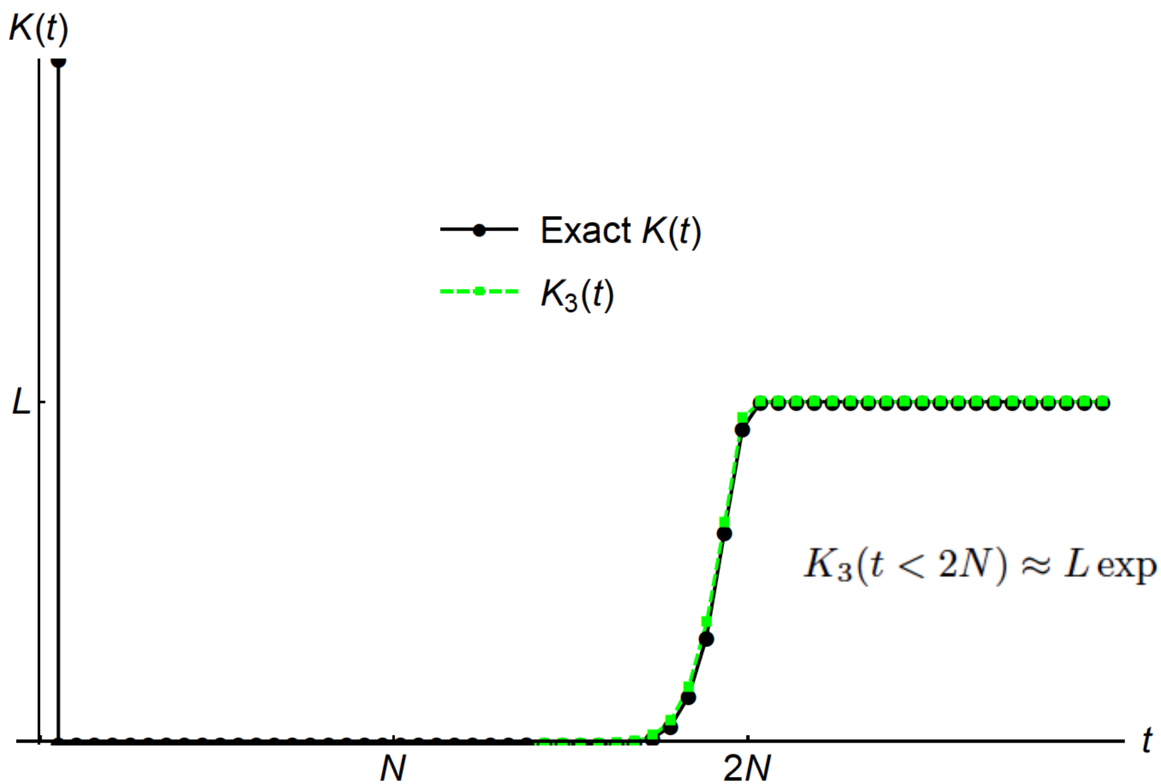


Energy scales:

- Single particle level spacing: N^{-1}
- Many-body level spacing: $\sim {}^{1}/_{L} = 2^{-N}$
- Plateau time: $t_* = 2N$ – residual effect from single particle level repulsion

Looking at the big (many-body) picture

The spectral form factor in the linear scale ($N = 240, L = 2^{240}$)



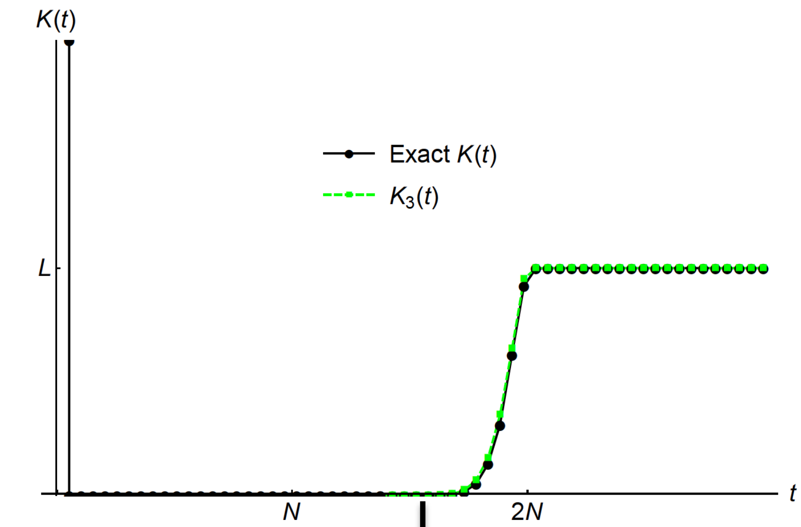
$$K_3(t < 2N) \approx L \exp\left[-\frac{(2N-t)^{\frac{3}{2}}}{3\pi N^{\frac{1}{2}}}\right]$$

Eventually (that is, as N grows further), all single-particle features are not even noticeable by a naked eye and a step function is an excellent fit:

$$K_{\mathrm{ramp}}(t) \approx L(1 - \Theta(2N\alpha - t))$$

$$\alpha = 1 - \tfrac{2}{3}\Gamma\left(\tfrac{2}{3}\right)(3\pi/N)^{\frac{2}{3}} + O(N^{-3/5})$$

"New" level statistics. Islands of level attraction

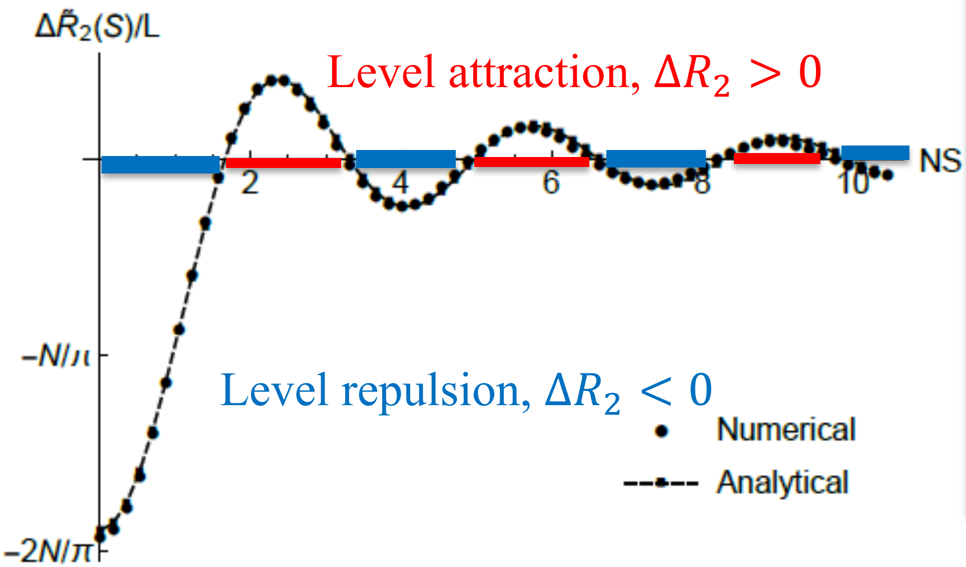


$$K_{\mathrm{ramp}}(t) \approx L(1 - \Theta(2N\alpha - t))$$

Fourier transform

$$\tilde{R}_2(S \sim N^{-1}) \approx \frac{L^2}{\sqrt{2\pi N}} - \frac{2N\alpha L}{\pi}\frac{\sin(2N\alpha S)}{2N\alpha S}$$

Fourier transform



We observe level "repulsion" term $\sim \frac{\sin(2NS)}{2NS}$, with smaller islands of level "attraction."

Compare with standard GUE statistics,

$$R_2(\varepsilon_1, \varepsilon_2) \approx \frac{N^2}{\pi^2}\left(1 - \frac{\sin^2(N(\varepsilon_1 - \varepsilon_2))}{N^2(\varepsilon_1 - \varepsilon_2)^2}\right)$$

A side comment: on level attraction…

J Stat Phys (2016) 163:998–1048
DOI 10.1007/s10955-016-1508-x

On Many-Body Localization for Quantum Spin Chains

John Z. Imbrie[1]

Abstract For a one-dimensional spin chain with random local interactions, we prove that many-body localization follows from a physically reasonable assumption that limits the amount of level attraction in the system. The construction uses a sequence of local unitary transformations to diagonalize the Hamiltonian and connect the exact many-body eigenfunctions to the original basis vectors.

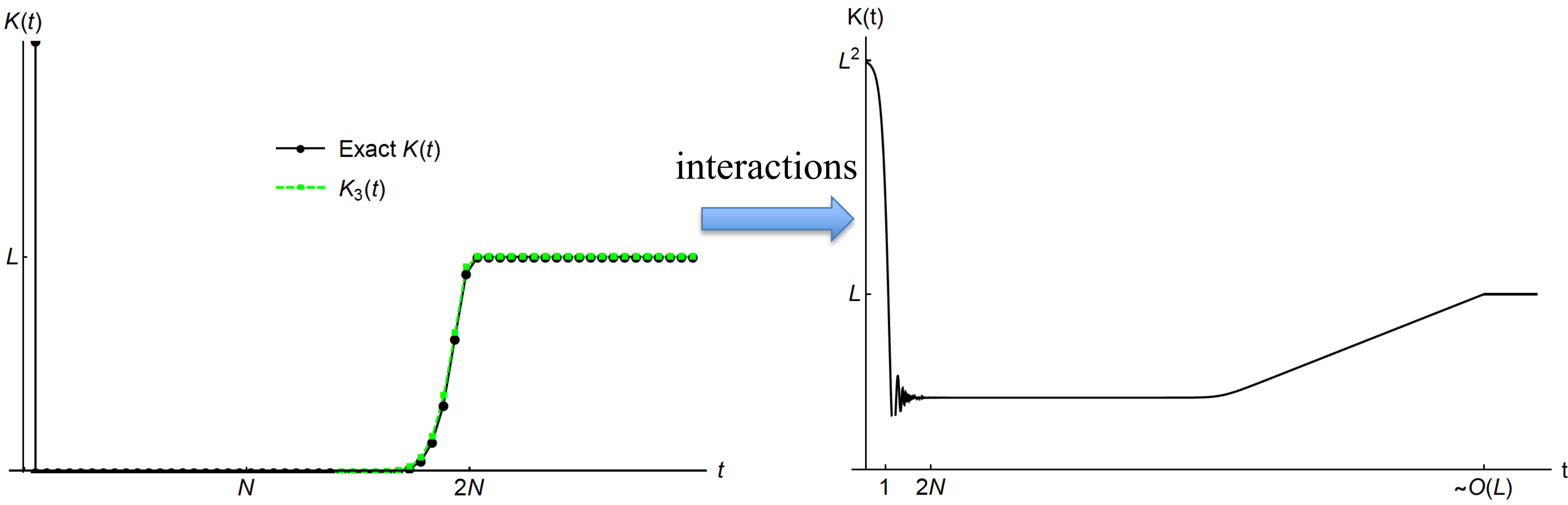
Assumption LLA(ν, C) (*Limited level attraction*) Consider the Hamiltonian H in a box Λ with $|\Lambda| = K + K' + 1 = n$. With the given probability distribution for $\{h_i, \Gamma_i, J_i\}$, its eigenvalues satisfy

$$P\left(\min_{\alpha\neq\beta} |E_\alpha - E_\beta| < \delta\right) \leq \delta^\nu C^n, \qquad (1.3)$$

for all $\delta > 0$ and all n.

Results presented so far have no direct relation to MBL (since we have no interactions), but if there is a crossover between the non-interacting chaotic fermion model and many-body quantum chaos, echoes of level attraction may survive (and the single-particle energy scales are enormous compared to many-particle level spacings). To be explored further…

Towards true many-body quantum chaos



Field theoretic (σ-model) approach

- A path integral formula for the SFF $K(t) = \langle |Z(it)|^2 \rangle = \sum_{n,m} \left\langle e^{i(E_n - E_m)t} \right\rangle$

$$K(t) = \left\langle \int \mathcal{D}(\bar{\psi}, \psi) \exp\left\{ i \sum_{\epsilon} \bar{\psi}_i^a(\epsilon) \left[\epsilon \delta_{ij} \delta_{ab} - h_{ij} \sigma^3_{ab} \right] \psi_j^b(\epsilon) \right\} \right\rangle_h$$

Saad, Shenker, and Stanford (2019)

- h_{ij} is the random matrix to be averaged over.
- a & $b = \pm 1$ correspond to forward, $Z(it)$/backward, $Z(-it)$, evolution.
- $\epsilon = 2\pi(n + 1/2)/t$ is the "Matsubara" energy.

- Averaging over Gaussian h-distribution generates "interaction" terms.
- In the σ −model, we decouple them again with a collective field, $Q_{ab}(\epsilon, \epsilon')$.

σ-model action

- The σ-model action in term of the Hermitian matrix, $Q_{ab}(\epsilon, \epsilon')$.

$$K(t) = \frac{\int \mathcal{D}Q \exp\left\{-\frac{N}{2}\mathrm{Tr}\,\hat{Q}^2 + N\,\mathrm{Tr}\left[-t\left(i\epsilon\delta_{\epsilon\epsilon'} + \hat{Q}\sigma^3\right)\right]\right\}}{\int \mathcal{D}Q e^{-\frac{N}{2}\mathrm{Tr}\,Q^2}}$$

- $\frac{\delta S}{\delta Q} = 0$ yields a series of saddle points, labeled by signs, **s** (c.f., Kamenev & Mezard)

$$\hat{Q}_{\mathrm{saddle}} = \hat{U}\hat{\Lambda}(\mathrm{s})\hat{U}^{-1} \qquad \Lambda_{ab}(\epsilon, \epsilon') = \delta_{ab}\delta_{\epsilon\epsilon'}\left[-ia\epsilon + \underbrace{s_a(\epsilon)}_{\pm 1}\sqrt{4-\epsilon^2}\right]$$

- The SFF is given by the sum over the saddles and fluctuations around them

$$K(t) = \sum_s e^{-S[\Lambda^{(s)}]} \frac{\int \mathcal{D}\delta Q \exp\left\{-\frac{N}{2}\sum \delta Q_{ab}(\varepsilon, \varepsilon')D^{-1}_{ab}(\varepsilon, \varepsilon')\delta Q_{ba}(\varepsilon', \varepsilon)\right\}}{\int \mathcal{D}Q e^{-\frac{N}{2}\mathrm{Tr}Q^2}}$$

Massive, soft, and zero modes

- The SFF is given by the sum over the saddles and fluctuations around them

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The saddle point action contributes to the slope.

- The fluctuation propagator (depending on a saddle) come in 3 varieties:
  - Massive modes, $D^{-1}_{ab}(0,0) = \text{const} \sim O(1)$
  - Soft modes ($q \to 0$ limit of "diffusons" in the Altshuler et al theory of metals)

    $$D^{-1}_{ab}(\epsilon,\epsilon') \propto (a\epsilon - b\epsilon')$$
  - and zero modes, where simply $D^{-1}_{ab}(0,0) = 0$

- Zero-mode contribution *Also by B. Swingle et al (tbp, 2020)*

$$\frac{\int_{\mathcal{V}} \mathcal{D}Q}{\int_{\mathcal{V}} \mathcal{D}Q \exp\left[-\frac{N}{2} Q_{ab}(\varepsilon,\varepsilon') Q_{ba}(\varepsilon',\varepsilon)\right]} \propto \frac{\mathcal{V}}{(1/\sqrt{N})^{\mathcal{N}_z}} \propto N^{4t/\pi}$$

Other modes modify the exp coefficient *(Liao, Vikram, VG 2020)*

c.f. Kamenev and Mezard (1999)

To make the disordered non-interacting fermions and SYK2 many-body chaotic, we can add weak <u>non-random</u> interactions to the models: ***SYK2+***

Interacting theory (work in progress)

- Interacting sigma model

HS field (decouples the interactions)

$$K(t) = \frac{1}{Z}\int \mathcal{D}\phi \exp\left\{\frac{i}{2}\int U^{-1}\phi\,\sigma^3\phi\right\}$$

$$\times \int \mathcal{D}Q \exp\left\{-\frac{\pi\nu}{4\tau_{\rm el}}\mathrm{Tr}Q^2 + \mathrm{Tr}\ln\left[-it\left(\omega_n\sigma^3 + \frac{\nabla^2}{2m} + \frac{1}{\sqrt{t}}\begin{bmatrix}\phi^L & 0\\ 0 & \phi^R\end{bmatrix} + \frac{i}{2\tau_{\rm el}}Q\right)\sigma^3\right]\right\}$$

- The soft modes (diffusons or $1/\omega$'s) now follow from a stochastic Diff. Eq.

similar to the standard Altshuler, Aronov, Khmelnitskii (1982) dephasing Eq. like

$$\left\{\partial_t - D\nabla^2 - i\left[\phi_+(\mathbf{r},t) - \phi_-(\mathbf{r},0)\right]\right\} D_{t,t'}(\mathbf{r},\mathbf{r}') = \delta(t-t')\delta(\mathbf{r}-\mathbf{r}')$$

<u>classical (thermal</u>) component of the H.S. field, ϕ, gives stochasticity & dephasing.

- Interacting sigma model for the interacting many-body SFF (ongoing work)

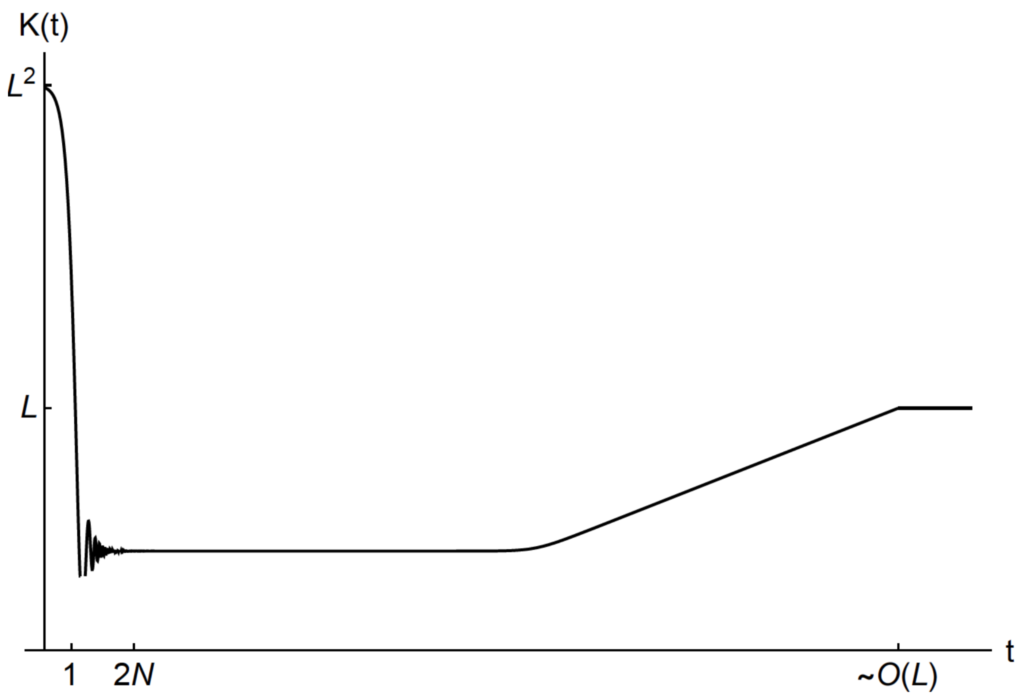
$$\frac{1}{Dk^2 + i\omega} \overset{?}{\Rightarrow} \frac{1}{Dk^2 + i\omega + \tau^{-1}}$$

Exponential ramp $\overset{?}{\Rightarrow}$ Linear WD ramp

Interacting theories with single-particle chaos?

- Chaos emerging from interactions – Wigner-Dyson statistics over finely-spaced many-body levels.

- Transition from single-particle to "true" many-body chaos should manifest itself as emergence of an approximately linear ramp at $t \sim L = 2^N$, overwhelming the single-particle features at $t \sim N$.



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